Bohmian Mechanics

Quantum Foundations · Module 6 — Interpretations

Node 6.4

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Interpretations **Node:** 6.4

Bohmian mechanics — due to Louis de Broglie (1927) in its pilot-wave form and rediscovered and developed by David Bohm (1952), with later refinement by Bell, Dürr, Goldstein, Zanghì, and others — is the canonical *hidden-variable* interpretation of quantum mechanics. It restores determinism and particle trajectories by supplementing the wavefunction with a second physical ingredient: actual particle positions. The wavefunction evolves under the Schrödinger equation exactly as in standard quantum mechanics and plays the role of a guiding field. Particles follow definite trajectories determined by the wavefunction through a first-order *guidance equation*. All quantum predictions are recovered, provided one assumes that particle positions are distributed according to $|\psi|^2$ — the **quantum equilibrium hypothesis**.

Bohmian mechanics is the interpretation one reaches by insisting that physics describes an observer-independent reality of point particles moving in space, at the cost of accepting that the dynamics is explicitly nonlocal.

Formal Statement

The two dynamical objects

A Bohmian universe is specified by:

1. A **wavefunction** $\psi(x_1, \ldots, x_N, t)$ on configuration space, evolving under the usual Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\nabla_k^2 + V(x_1, \ldots, x_N)\right)\psi.$$

2. **Actual positions** $Q_1(t), \ldots, Q_N(t)$ of N particles, evolving under the **guidance equation**:

$$\frac{dQ_k}{dt} = \frac{\hbar}{m_k}\,\mathrm{Im}\,\frac{\nabla_k\psi}{\psi}\bigg|_{x_1=Q_1,\ldots,x_N=Q_N}.$$

The wavefunction guides the particles; the particles do not back-react on the wavefunction. The theory is first-order in time (like a flow, not like Newtonian mechanics), and it is fully deterministic: given ψ and the initial positions $Q_k(0)$, the future evolution of everything is fixed.

Quantum equilibrium hypothesis

The statistical content of the theory comes from a single assumption about initial conditions: the positions of particles in any subsystem are distributed according to $|\psi|^2$ at some time. The continuity equation

$$\frac{\partial|\psi|^2}{\partial t} + \sum_k \nabla_k \cdot \left(|\psi|^2\, v_k\right) = 0, \qquad v_k = \frac{\hbar}{m_k}\,\mathrm{Im}\,\frac{\nabla_k\psi}{\psi},$$

ensures that if $|\psi|^2$ holds initially, it holds for all time. This is called **equivariance**: the Bohmian flow carries the $|\psi|^2$ distribution into itself. The quantum equilibrium hypothesis is then the statement that our universe is in such an equilibrium.

Under quantum equilibrium, all empirical predictions of Bohmian mechanics agree exactly with those of standard quantum mechanics, because measurement outcomes — being reducible to final particle positions (pointer readings) — are distributed according to $|\psi|^2$. Non-equilibrium Bohmian mechanics, in which particle distributions differ from $|\psi|^2$, is a logically consistent possibility but is not observed (Valentini has argued it could leave cosmological signatures).

No separate measurement postulate

Bohmian mechanics has *no projection postulate.* Measurement is simply an interaction between a system and an apparatus in which the apparatus's pointer (a collection of Bohmian particles) ends up at one of several spatial locations. The Born rule is not postulated — it is *derived* from the guidance equation plus quantum equilibrium.

Intuition

The animating image is simple. Particles are real objects with definite locations at all times, moving through space along smooth trajectories. The wavefunction is a real field on configuration space that tells each particle how to move. A measurement outcome is just the final location of a pointer particle; there is nothing mysterious about it.

This is the interpretation a seventeenth-century natural philosopher would have wanted. It restores the classical picture almost in full: there are particles, they move, the trajectories exist, determinism holds. The only non-classical element is the nature of the guiding field, which lives on the $3N$-dimensional configuration space rather than on ordinary three-dimensional space and therefore couples distant particles instantaneously.

This instantaneous coupling is the price. Bell's theorem (Node 10.1) proves that any theory assigning definite values to measurement outcomes and reproducing quantum statistics must be nonlocal. Bohmian mechanics does not evade this — it *embraces* it. The guidance equation for particle k depends on the positions of all other particles simultaneously, so a measurement on one half of an entangled pair instantaneously alters the guidance of the other half. There is no signaling (the $|\psi|^2$ distribution prevents any observable faster-than-light effects), but the underlying dynamics is explicitly nonlocal in a way that sits uneasily with special relativity.

The second cost is mathematical elegance. The Schrödinger equation plus guidance equation is not the most economical description — it has two dynamical entities where standard quantum mechanics has one. And the quantum equilibrium hypothesis must be added as an independent postulate (though it has a statistical-mechanics-style justification in terms of typicality arguments).

The advantage is a clean ontology. Bohmian mechanics says exactly what exists (particles and wavefunction), exactly where it is (positions in configuration space), and exactly how it evolves (Schrödinger + guidance). There is no measurement problem because measurement is just a particular kind of interaction. There is no preferred-basis problem because the preferred basis is position, by fiat. There is no observer problem because observers are collections of Bohmian particles like any other.

Structural Role

Bohmian mechanics plays several distinctive structural roles in the foundations literature:

- **Existence proof for hidden variables.** Before Bohm, it was widely believed (following von Neumann's flawed 1932 "proof") that hidden-variable completions of quantum mechanics were impossible. Bohmian mechanics refuted this by exhibiting one. The von Neumann argument was shown by Bell to rest on an unjustified assumption (linearity of hidden-variable value assignments over non-commuting observables).

- **Counterexample to naïve locality claims.** Bohmian mechanics makes it concrete that nonlocal hidden-variable theories can reproduce quantum statistics. Bell's theorem was developed in part by thinking about what Bohmian nonlocality actually requires, and the Bell inequalities are tight: any theory reproducing quantum statistics and assigning definite outcomes must have this much nonlocality.
- **Position as the fundamental observable.** Bohmian mechanics takes position as the primitive ontology and derives the other observables' behavior from measurement dynamics. This makes it contextual in Bell's sense (Node 10.2): the "value" of, say, spin is not a property of the particle but of the joint system of particle plus measuring apparatus. This squares Bohmian mechanics with Kochen–Specker contextuality by accepting it straightforwardly.
- **ψ-ontic commitment.** The wavefunction in Bohmian mechanics is a real physical field, not an epistemic tool. This makes Bohmian mechanics strongly ψ-ontic, and it passes the PBR theorem (Node 10.3) trivially.
- **Contrast class to many-worlds.** Both Bohmian mechanics and many-worlds drop the projection postulate. Many-worlds keeps only the wavefunction; Bohmian mechanics keeps the wavefunction *and* adds particles. Defenders of many-worlds argue that Bohmian particles are redundant (the wavefunction is doing all the work); defenders of Bohmian mechanics argue that the particles are the only thing preventing the wavefunction from being a sum of unrealized possibilities.

Mathematical Development

Deriving the guidance equation

Write the wavefunction in polar form $\psi = Re^{iS/\hbar}$ with R, S real. Substituting into the Schrödinger equation and separating real and imaginary parts yields two equations:

$$\frac{\partial R^2}{\partial t} + \sum_k \nabla_k \cdot \left(R^2 \frac{\nabla_k S}{m_k} \right) = 0,$$

$$\frac{\partial S}{\partial t} + \sum_k \frac{(\nabla_k S)^2}{2m_k} + V + Q = 0, \qquad Q = -\sum_k \frac{\hbar^2}{2m_k} \frac{\nabla_k^2 R}{R}.$$

The first equation is a continuity equation for the density $R^2 = |\psi|^2$, with velocity field $v_k = \nabla_k S/m_k$. The second is a Hamilton–Jacobi equation for S supplemented by an additional term Q, the **quantum potential**, which is the only place $\hbar$ appears. Bohm's original 1952 formulation was cast in these terms: particles follow Newtonian-like trajectories under the combined potential $V + Q$.

The modern formulation (Dürr–Goldstein–Zanghì) drops the quantum potential and simply reads off the guidance equation directly from the probability current:

$$j_k = \frac{\hbar}{m_k} \operatorname{Im}(\psi^* \nabla_k \psi) = |\psi|^2 \frac{\nabla_k S}{m_k} = |\psi|^2 v_k,$$

giving

$$v_k = \frac{j_k}{|\psi|^2} = \frac{\hbar}{m_k} \operatorname{Im} \frac{\nabla_k \psi}{\psi}.$$

This form makes no reference to S (which is only defined up to $2\pi\hbar$ at nodes of ψ) and generalizes cleanly to spinors and many-particle systems.

Equivariance

If the distribution of particle positions is $\rho(x,t)$, then ρ satisfies a continuity equation:

$$\frac{\partial \rho}{\partial t} + \sum_k \nabla_k \cdot (\rho v_k) = 0.$$

Since $|\psi|^2$ also satisfies this equation with the same velocity field (from the Schrödinger equation), any distribution equal to $|\psi|^2$ at one time remains equal to $|\psi|^2$ at all later times. This is *equivariance*, and it is the technical basis for the quantum equilibrium hypothesis being consistent.

Quantum equilibrium and typicality

The question "why are particles distributed according to $|\psi|^2$?" has been addressed by Dürr–Goldstein–Zanghì through a **typicality argument**, analogous to Boltzmann's justification of the uniform distribution on phase space in classical statistical mechanics. The argument is: among the measure-$|\Psi|^2$ set of possible initial particle configurations of the universe, the vast majority of initial conditions lead to subsystem distributions that are empirically indistinguishable from $|\psi|^2$. So quantum equilibrium is *typical* in a precise measure-theoretic sense, even if it is not inevitable.

Nonlocality in entangled systems

For a two-particle entangled state $\psi(x_1, x_2)$, the guidance equation for particle 1 depends on x_2:

$$\dot{Q}_1 = \frac{\hbar}{m_1} \operatorname{Im} \left. \frac{\nabla_1 \psi(x_1, x_2)}{\psi(x_1, x_2)} \right|_{x_1 = Q_1, x_2 = Q_2}.$$

If ψ does not factorize, the velocity of particle 1 depends on where particle 2 is *right now.* Measuring particle 2 (which in Bohmian mechanics means causing its position to be correlated with a macroscopic pointer) alters the effective wavefunction that guides particle 1, and this alteration propagates instantaneously.

This is the explicit nonlocality Bell's theorem requires of any definite-outcome theory. The theory is compatible with no-signaling because the $|\psi|^2$ distribution over hidden initial conditions averages out the nonlocal influences at the statistical level — any experimenter making only macroscopic measurements cannot use the nonlocal dynamics to send information.

Measurement as entanglement plus position readout

Consider a spin measurement in the Bohmian picture. The spin-1/2 particle has a wavefunction $\psi(x) = \alpha\phi_+(x) + \beta\phi_-(x)$ where $\phi_\pm$ are spin eigenstates with spatial wavepackets. A Stern–Gerlach magnet couples spin to position, producing after evolution a bipartite wavefunction in which spin-up packets are at position $+z$ and spin-down packets are at position $-z$:

$$\psi(x,t) = \alpha\phi_+(x - z_0\hat{z}, t) + \beta\phi_-(x + z_0\hat{z}, t).$$

The actual Bohmian particle ends up in one of the two packets — whichever one contains its initial position. Which one it is depends deterministically on the initial conditions, but because those conditions are distributed according to $|\psi|^2$, the observed statistics are $|\alpha|^2$ and $|\beta|^2$. The spin "value" is not a property of the particle; it is a feature of the apparatus–particle interaction that happens to correlate cleanly with the final position.

Worked Example

Double-slit with Bohmian trajectories

A particle impinges on a screen with two slits. The wavefunction after the slits is the superposition $\psi = \psi_1 + \psi_2$ of two outgoing wavepackets. Downstream, the two packets overlap and interfere, producing the standard $\cos^2$ fringe pattern in $|\psi|^2$.

In Bohmian mechanics, the particle has a definite position at all times. It goes through one slit — slit 1 if its initial position is in the left half of the configuration space, slit 2 otherwise. After the slits, the particle's trajectory is guided by the total ψ, and in the overlap region the trajectories become strongly curved, bunching toward the bright fringes and avoiding the dark fringes.

Numerical computations of these trajectories (originally by Philippidis, Dewdney, and Hiley in 1979) show a distinctive "no-crossing" property: Bohmian trajectories never cross each other in configuration space. This means that in the symmetric double-slit, a particle going through the left slit always ends up on the left half of the screen, and vice versa. (Contrast with the Feynman picture, where paths cross freely.)

The statistical distribution of impact positions is $|\psi|^2$, reproducing the interference pattern exactly. An individual particle's path is a definite curve, but which curve depends on the initial position, and the ensemble of curves over the $|\psi|^2$ distribution produces interference.

Crucially, *which slit the particle went through* is not observable without disturbing the wavefunction. Any detector at one slit destroys the coherence of the two packets and thus destroys the interference. Bohmian mechanics is consistent with the standard experimental phenomenology: in the interference case, the question "which slit did it go through?" has an *answer* according to the theory (one can in principle compute it from initial conditions), but no *measurement* can reveal that answer without erasing the interference. This is the sense in which Bohmian hidden variables are "hidden."

EPR in Bohmian mechanics

Consider an entangled singlet pair $|\psi\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$. Alice measures her particle's spin along some axis; Bob measures his along another.

In Bohmian mechanics, both particles have definite position trajectories throughout. When Alice performs her measurement, her apparatus couples to her particle's spin-position correlation, and her particle ends up in one of the spin-up or spin-down packets. The *wavefunction* is still an entangled two-particle wavefunction, so the "effective wavefunction" guiding Bob's particle is instantaneously updated: Bob's particle is now guided by the conditional wavefunction corresponding to Alice's outcome.

This update is nonlocal in the strict sense — it happens at Bob's location at the same time as Alice's measurement, regardless of spatial separation. But the statistics of Bob's subsequent measurement remain 1/2, 1/2 if he hasn't yet been informed of Alice's outcome, so no signal has been sent. The nonlocality is "under the hood": visible in the equations of motion of hidden variables but invisible at the level of observed statistics.

Bell's theorem is what *forces* this nonlocality: if Bohmian mechanics were local, it could not reproduce the violations of CHSH observed in real EPR experiments. The theorem guarantees that any deterministic hidden-variable theory with definite outcomes must have this feature. Bohmian mechanics has it explicitly and unapologetically.

Hydrogen ground state

The hydrogen ground-state wavefunction $\psi = (\pi a_0^3)^{-1/2} e^{-r/a_0}$ is real. Since the guidance equation is proportional to $\mathrm{Im}(\nabla\psi/\psi)$, and the wavefunction is real, the velocity field vanishes identically. Therefore the Bohmian electron in the ground state is *at rest*: it sits at some definite position in the cloud and does not move at all.

This is unintuitive from a classical standpoint — a stationary electron in a hydrogen atom? — but consistent. The orbital picture of an electron orbiting the nucleus does not correspond to Bohmian reality. The "motion" in the classical picture is really the spread of the wavefunction; the Bohmian particle itself is stationary in s-states. Kinetic energy appears through the quantum potential Q, not through actual motion. This is one of the places where Bohmian mechanics' ontology diverges most strikingly from classical intuition.

Cross-Links

- **Module 6.1 (Measurement problem):** Bohmian mechanics resolves the measurement problem by eliminating the collapse postulate and treating measurement as an interaction that correlates the apparatus position with the system state.
- **Module 6.3 (Many-worlds):** Both drop the projection postulate. Many-worlds keeps only the wavefunction; Bohmian mechanics adds particles. Structurally, Bohmian mechanics can be seen as "many-worlds with a marker pointing to the branch we're in."
- **Module 10.1 (Bell's theorem):** Forces Bohmian mechanics to be nonlocal. The nonlocality is visible in the guidance equation for entangled states.
- **Module 10.2 (Kochen–Specker):** Bohmian mechanics is contextual: the "value" of a spin measurement depends on the apparatus, not just on the particle. This is consistent with KS, which forbids non-contextual hidden variables.
- **Module 10.3 (PBR theorem):** Bohmian mechanics is ψ-ontic (the wavefunction is real) and so survives PBR.
- **Classical mechanics:** The guidance equation is first-order in time, like a classical flow, and the quantum potential plays a role analogous to a Hamilton–Jacobi potential.
- **Statistics:** Quantum equilibrium is analogous to thermodynamic equilibrium in statistical mechanics — a typical distribution that dominates observations even though non-equilibrium initial conditions are logically possible.

Structural Compression

Invariant: Particles have definite positions at all times, guided by a wavefunction that evolves unitarily. Measurement reveals position; other "values" (spin, momentum) are contextual features of the particle–apparatus interaction. Quantum statistics are recovered because positions are distributed according to $|\psi|^2$.

Mechanism: Schrödinger equation for ψ, guidance equation for particle positions, quantum equilibrium hypothesis $\rho = |\psi|^2$ as the initial condition.

Cross-System Echo: Latent-variable models in statistics, hidden Markov state in Bayesian filtering, and stochastic particle methods in fluid dynamics all share the pattern "guiding field + tracer particle distributed by that field's density."

Exercises

1. Derive the guidance equation from the Schrödinger equation via the polar decomposition $\psi = Re^{iS/\hbar}$. Verify that the velocity field is $\nabla S/m$ and that $|\psi|^2$ satisfies a continuity equation with this velocity.
2. Show that the hydrogen ground-state wavefunction, being real, yields zero Bohmian velocity. Discuss the tension with the classical picture of an orbiting electron.
3. For a particle in a one-dimensional box in the n-th stationary state, compute the Bohmian trajectories. Verify that the particle is at rest, and explain how this is consistent with nonzero kinetic energy expectation values.
4. Construct the Bohmian trajectories for a Gaussian wavepacket in free space. Show that the trajectories spread linearly with time, matching the $|\psi|^2$ spread.

5. For an entangled singlet pair, write the guidance equation for each particle and show explicitly that it depends on the other particle's position. Use this to explain how Bohmian mechanics is nonlocal yet no-signaling.
6. Discuss the quantum equilibrium hypothesis. What would it mean for the universe to *not* be in quantum equilibrium, and what observable signatures would such a state have?
7. Argue, structurally, why Bohmian mechanics accepts the results of the Kochen–Specker theorem as a feature rather than a problem. What does contextuality mean in the Bohmian framework?

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Concepts: dynamical-systems, state-space, probability-space
Prerequisites: 6.1
Enables: 6.6