Qubits and Quantum Registers

Quantum Foundations · Module 7 — Quantum Information

Node 7.1

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Quantum Information **Node:** 7.1

This is the foundation node of the quantum information module. The qubit is the simplest quantum system — and the universal building block for quantum computation, communication, and cryptography. Every protocol in the rest of this module is built on registers of qubits.

Formal Definition

A **qubit** is a quantum system with state space $\mathcal{H} = \mathbb{C}^2$, equipped with a distinguished orthonormal basis $\{|0\rangle, |1\rangle\}$ called the **computational basis**.

A general qubit state is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad \alpha, \beta \in \mathbb{C}, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

A **quantum register** of n qubits is a system with state space

$$\mathcal{H}^{\otimes n} = (\mathbb{C}^2)^{\otimes n} = \mathbb{C}^{2^n},$$

with computational basis

$$\{|x\rangle : x \in \{0,1\}^n\}$$

where $|x\rangle$ is shorthand for $|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$.

A general n-qubit state is

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} c_x |x\rangle, \qquad \sum_x |c_x|^2 = 1,$$

with 2^n complex amplitudes (and $2^n - 1$ independent real parameters after normalization and global phase).

Intuition

A qubit is two classical bits' worth of "answers" entangled with one bit's worth of "information you can read." It is *not* a continuous classical bit; it is *not* a probabilistic classical bit; it is a structurally distinct object whose reduction-to-classical only happens at the moment of measurement.

The exponential dimensionality of registers — 2^n amplitudes for n qubits — is the structural source of quantum computational resources. A 50-qubit register has more amplitudes than there are atoms in the observable universe; this is the structural fact that quantum simulation exploits.

Structural Role

This node is the operational entry point to all quantum information protocols:

- No-cloning theorem (7.2) is a constraint on what can be done to qubit registers.
- Quantum channels (7.3) are the dynamics on density matrices on $\mathcal{H}^{\otimes n}$.
- Entropy and mutual information (7.4) measure resources on quantum registers.
- Teleportation and superdense coding (7.6) are protocols on small registers (2-3 qubits).

It is also the entry point to VQA: parameterized quantum circuits act on registers of this form.

Mathematical Development

Single-qubit gates

Single-qubit operations are unitaries on $\mathbb{C}^2$ — equivalently, elements of $SU(2)$ (up to phase). The Pauli operators X, Y, Z generate the Pauli group; the Hadamard H, phase S, and T gates generate the Clifford+T set, which is universal for single-qubit unitaries (approximately).

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Two-qubit gates

The CNOT (controlled-NOT) gate is the canonical two-qubit entangling gate:

$$\text{CNOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X.$$

The set $\{H, S, \text{CNOT}\}$ plus any non-Clifford single-qubit gate (e.g., T) is **universal** for quantum computation: any unitary on n qubits can be approximated to arbitrary precision by a circuit of these gates.

Quantum registers vs classical registers

Property	Classical n-bit register	Quantum n-qubit register
State space	$\{0,1\}^n$, 2^n states	$\mathbb{C}^{2^n}$, $2 \cdot 2^n - 2$ real parameters
Operations	Boolean functions	Unitary operators
Read-out	Deterministic	Probabilistic (Born rule)
Composition	Cartesian product	Tensor product
Information content	n bits	Up to n qubits' worth of entanglement, but only n bits per measurement

The last row is Holevo's theorem (Node 7.5): you cannot extract more than n classical bits from n qubits, despite the exponentially larger state space.

Worked Example

Creating a Bell state

Starting from $|00\rangle$:

1. Apply H to qubit 1: $H \otimes I|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$.
2. Apply CNOT (control = qubit 1, target = qubit 2): $\text{CNOT}\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle$.

Two gates, one Bell state. This is the simplest non-trivial quantum protocol — and it is the entry point to teleportation, superdense coding, QKD, and Bell tests.

Cross-Links

- **Module 2:** State postulate (2.1) and tensor products (2.5) are the foundations.
- **Module 4:** Bell states (4.4) are the canonical entangled qubit states.
- **VQA:** All variational circuits are built from gates on qubit registers.
- **Linear Algebra:** The whole machinery of unitary matrices on $\mathbb{C}^{2^n}$.
- **Statistics:** Measurement of n qubits in the computational basis returns a sample from a distribution on $\{0, 1\}^n$.

Structural Compression

Invariant: Qubit register dimension scales as 2^n; operations are unitaries; readouts are samples from a Born-rule distribution.

Mechanism: Tensor product of n copies of $\mathbb{C}^2$.

Cross-System Echo: Classical bit registers grow linearly in storage; quantum registers grow exponentially in state space dimension. The gap is the structural source of quantum advantage.

Exercises

1. Show that any single-qubit unitary can be written as $U = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)$ (Z-Y-Z decomposition).
2. Verify that the CNOT gate maps $|+\rangle|0\rangle \to \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
3. Show that an n-qubit pure state has $2^{n+1} - 2$ real parameters before normalization, and $2 \cdot 2^n - 2$ after global-phase removal.
4. Construct a 3-qubit GHZ state $(|000\rangle + |111\rangle)/\sqrt{2}$ using H and CNOT gates.
5. Argue why the exponential state-space dimension does *not* automatically translate to exponential computational speedup. (Hint: read-out is constrained by Holevo's bound, Node 7.5.)

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Concepts: hilbert-space, tensor-product, superposition
Prerequisites: 2.1, 2.5
Enables: 7.2, 7.3

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