Quantum Channels and CPTP Maps

Quantum Foundations · Module 7 — Quantum Information

Node 7.3

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Quantum Information **Node:** 7.3

A **quantum channel** is the most general physically realizable linear operation on quantum states. It generalizes unitary evolution (which describes closed systems) to arbitrary open-system processes that include noise, decoherence, measurement, and discarding of subsystems. The class of quantum channels coincides with the class of **completely positive trace-preserving (CPTP) maps**, and this characterization is one of the central results of quantum information theory.

Three equivalent representations of quantum channels — Stinespring dilation, Kraus operator sum, and the Choi matrix — give complementary views of the same mathematical object. Together they form the standard toolkit for analyzing any quantum process in which the system is not perfectly isolated. Every result in the subsequent modules about decoherence, measurement, noise, or quantum information flow is ultimately a statement about channels.

Formal Definition

CPTP maps

Let $\mathcal{B}(\mathcal{H})$ denote the space of bounded operators on a Hilbert space $\mathcal{H}$. A **quantum channel** is a linear map

$$\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$$

satisfying two conditions:

1. **Trace preservation (TP):** For every $\rho \in \mathcal{B}(\mathcal{H}_A)$,

   $$\mathrm{Tr}(\mathcal{E}(\rho)) = \mathrm{Tr}(\rho).$$

   This ensures that probabilities sum to 1 after the channel acts.

2. **Complete positivity (CP):** For every ancillary Hilbert space $\mathcal{H}_R$ of any dimension, the extension $\mathcal{E} \otimes \mathrm{id}_R$ maps positive operators on $\mathcal{H}_A \otimes \mathcal{H}_R$ to positive operators on $\mathcal{H}_B \otimes \mathcal{H}_R$:

   $$X \geq 0 \implies (\mathcal{E} \otimes \mathrm{id}_R)(X) \geq 0.$$

Complete positivity is strictly stronger than ordinary positivity (which demands only $\mathcal{E}(\rho) \geq 0$ for $\rho \geq 0$). The standard counterexample is the transpose map $T : \rho \mapsto \rho^T$, which is positive but not completely positive — when tensored with the identity and applied to an entangled state, it can produce negative eigenvalues (this is the basis of the Peres PPT criterion in Node 4.2).

Stinespring dilation

Theorem (Stinespring 1955). A linear map $\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is CPTP if and only if there exist an environment Hilbert space $\mathcal{H}_E$, a pure state $|0\rangle_E \in \mathcal{H}_E$, and an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\mathcal{E}(\rho) = \mathrm{Tr}_E \left(V \rho V^\dagger \right).$$

Equivalently, every channel can be realized as a unitary on a larger system (system + environment) followed by tracing out the environment. The environment can always be chosen with dimension at most $\dim(\mathcal{H}_A) \cdot \dim(\mathcal{H}_B)$.

Kraus representation

Theorem (Kraus 1983). A linear map $\mathcal{E}$ is CPTP if and only if there exist operators $\{K_k\}_{k=1}^N$, $K_k : \mathcal{H}_A \to \mathcal{H}_B$, with

$$\sum_k K_k^\dagger K_k = I_A$$

(completeness condition, equivalent to trace preservation), such that

$$\mathcal{E}(\rho) = \sum_k K_k \rho K_k^\dagger.$$

The K_k are called **Kraus operators** (or operation elements). The representation is not unique — different Kraus sets can give the same channel, and the number of Kraus operators can be taken $\leq \dim(\mathcal{H}_A) \cdot \dim(\mathcal{H}_B)$.

Choi matrix

The **Choi matrix** of a channel $\mathcal{E}$ is the operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ obtained by applying $\mathcal{E}$ to one half of a maximally entangled state:

$$C_\mathcal{E} = (\mathrm{id}_A \otimes \mathcal{E})(|\Omega\rangle\langle\Omega|), \qquad |\Omega\rangle = \sum_i |i\rangle_A \otimes |i\rangle_A.$$

By the **Choi–Jamiołkowski isomorphism**, $\mathcal{E}$ is CPTP if and only if $C_\mathcal{E} \geq 0$ and $\mathrm{Tr}_B(C_\mathcal{E}) = I_A$. The Choi matrix provides a finite-dimensional representation of the channel and is the basis for many numerical approaches in quantum information.

Intuition

A quantum channel is "anything physical that takes quantum states in and produces quantum states out." The defining properties are natural:

- Linearity: if ρ is a probabilistic mixture $p_1\rho_1 + p_2\rho_2$, the output must be the same mixture $p_1\mathcal{E}(\rho_1) + p_2\mathcal{E}(\rho_2)$. Otherwise the channel would "know" the decomposition, which is unphysical.
- Trace preservation: probabilities must add up to 1.
- Positivity: outputs must be valid density operators.
- Complete positivity: even if the input system is part of a larger entangled system, the channel acting on just one half must still produce a valid joint state. Without this, channels could produce "negative probabilities" on entangled inputs, which would be unphysical.

The three representations each emphasize a different aspect:

- **Stinespring** says: channels are just unitaries on bigger systems, then throwing away part. This is the physicist's picture — every "noisy" or "lossy" process in the lab is actually a clean unitary interaction with an environment we don't control.
- **Kraus** says: channels are sums of "branches," each weighted by an operator. This is useful for explicit calculations and for thinking of channels as generalized measurements (each Kraus operator corresponds to a possible measurement outcome, if the environment is measured).
- **Choi** says: channels are operators on a doubled Hilbert space. This is the most mathematically compact view and turns questions about channels into questions about fixed density matrices.

The key structural fact — that the CPTP class exactly coincides with physically realizable operations, independent of how one defines "physically realizable" — is a deep theorem. It says: there is no ambiguity about what quantum operations can be done in principle. The mathematical definition captures physics exactly.

Structural Role

- **Unification of dynamics.** Unitary evolution, measurement (via generalized POVMs), discarding a subsystem (partial trace), noise, decoherence, and quantum error correction are all quantum channels. The structural payoff is that one framework handles all of them uniformly.
- **Foundation for open systems.** Module 8 (Lindblad equation) will introduce the continuous-time generator of a semigroup of channels. The Lindblad equation is to quantum channels what the master equation is to classical Markov processes.
- **Entropic bounds.** The data-processing inequality (Module 7.4) states that mutual information cannot increase under channel action: $I(A;B)_{(\mathrm{id}\otimes\mathcal{E})(\rho)} \le I(A;B)_\rho$. This is the quantum analogue of the classical fact that noisy channels destroy information, and is central to quantum Shannon theory.
- **Noise models for real hardware.** Variational quantum circuits run on noisy quantum devices are modeled by inserting a noise channel after each gate. The choice of channel (depolarizing, amplitude damping, dephasing, etc.) determines the error model for quantum error correction and fault tolerance.
- **Capacity theorems.** Quantum channels have several inequivalent capacities: classical capacity, quantum capacity, private capacity, entanglement-assisted capacity. These are all rates at which specific resources can be transmitted through the channel, and they form the core of quantum Shannon theory.

Mathematical Development

Equivalence of representations

Start from Stinespring: $\mathcal{E}(\rho) = \mathrm{Tr}_E(V\rho V^\dagger)$. Choose an orthonormal basis $\{|e_k\rangle\}$ of $\mathcal{H}_E$ and define

$$K_k = \langle e_k|_E V : \mathcal{H}_A \to \mathcal{H}_B.$$

Then

$$\mathrm{Tr}_E(V\rho V^\dagger) = \sum_k \langle e_k|V\rho V^\dagger|e_k\rangle = \sum_k K_k \rho K_k^\dagger,$$

and the completeness relation $V^\dagger V = I_A$ translates to $\sum_k K_k^\dagger K_k = I_A$. This gives the Kraus form.

Conversely, given Kraus operators $\{K_k\}$, one can construct an isometry $V|\psi\rangle = \sum_k K_k|\psi\rangle \otimes |e_k\rangle$, which is exactly a Stinespring dilation. So the two representations are equivalent.

The Choi matrix can be read off either: from Kraus form,

$$C_{\mathcal{E}} = \sum_k (I \otimes K_k)|\Omega\rangle\langle\Omega|(I \otimes K_k^\dagger).$$

Its positivity (since each term is a rank-1 positive operator) reflects complete positivity of $\mathcal{E}$.

Complete positivity versus positivity: the transpose map

Consider the transpose map $T : \rho \mapsto \rho^T$ on a single qubit. T is positive: if $\rho \geq 0$ then $\rho^T \geq 0$ (since they have the same spectrum). But $T \otimes \mathrm{id}$ applied to a Bell state $|\Phi^+\rangle\langle\Phi^+|$ on two qubits gives

$$(T \otimes I)|\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}\sum_{i,j} |i\rangle\langle j|^T \otimes |i\rangle\langle j| = \frac{1}{2}\sum_{i,j} |j\rangle\langle i| \otimes |i\rangle\langle j| = \frac{1}{2}\mathrm{SWAP},$$

which has eigenvalues $+1/2, +1/2, +1/2, -1/2$ — a negative eigenvalue. So T is positive but not completely positive, and therefore not a quantum channel.

This is exactly the mathematical obstruction exploited by the PPT (Peres–Horodecki) separability criterion in Node 4.2: applying the partial transpose to an entangled state can produce a non-positive matrix, detecting the entanglement.

Channel composition

If $\mathcal{E}_1 : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ and $\mathcal{E}_2 : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_C)$ are channels, their composition $\mathcal{E}_2 \circ \mathcal{E}_1$ is also a channel, with Kraus operators given by products:

$$(K_k^{(2)} K_l^{(1)}) \in \{\text{Kraus of } \mathcal{E}_2 \circ \mathcal{E}_1\}.$$

Channels form a monoid under composition with the identity channel.

Unital vs non-unital channels

A channel $\mathcal{E}$ is **unital** if $\mathcal{E}(I) = I$. Equivalently, the maximally mixed state is a fixed point. Examples: unitary channels, Pauli channels, depolarizing channels. Non-unital examples: amplitude damping (which drives toward $|0\rangle\langle 0|$ rather than the maximally mixed state).

Unital channels form a subclass with many nice properties: they are the quantum analogue of doubly-stochastic classical channels (which preserve the uniform distribution) and are the channels for which entropy is non-decreasing.

Worked Example

Depolarizing channel

The depolarizing channel on a qubit with parameter $p \in [0, 1]$ is

$$\mathcal{E}_p(\rho) = (1 - p)\rho + p \cdot \frac{I}{2}.$$

It leaves the state unchanged with probability $1 - p$ and replaces it with the maximally mixed state with probability p.

Kraus form: Using the Pauli operators,

$$\mathcal{E}_p(\rho) = \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}\big(X\rho X + Y\rho Y + Z\rho Z\big),$$

so the Kraus operators are

$$K_0 = \sqrt{1 - \tfrac{3p}{4}}\, I, \quad K_1 = \sqrt{\tfrac{p}{4}}\, X, \quad K_2 = \sqrt{\tfrac{p}{4}}\, Y, \quad K_3 = \sqrt{\tfrac{p}{4}}\, Z.$$

Check: $\sum_k K_k^\dagger K_k = (1 - 3p/4)I + 3(p/4)I = I$. Trace-preserving.

Action on Bloch vector: $\mathcal{E}_p$ shrinks the Bloch vector by $(1-p)$, mapping the Bloch sphere to a smaller ball of radius $1-p$ centered at the origin. As $p \to 1$, all states collapse to the maximally mixed state. This is the canonical model of "isotropic noise."

Amplitude damping channel

The amplitude damping channel models spontaneous emission: a qubit in the excited state $|1\rangle$ decays to the ground state $|0\rangle$ with probability γ per unit time. The Kraus operators are

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}.$$

The physical picture: K_0 is "no decay occurred" (the amplitude on $|1\rangle$ shrinks), K_1 is "decay occurred" (the qubit jumps from $|1\rangle$ to $|0\rangle$).

Trace-preservation check: $K_0^\dagger K_0 + K_1^\dagger K_1 = \mathrm{diag}(1, 1-\gamma) + \mathrm{diag}(0, \gamma) = I$. ✓

Effect on a state: Starting from $\rho = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix}$, one finds

$$\mathcal{E}(\rho) = \begin{pmatrix} a + \gamma(1-a) & \sqrt{1-\gamma}\, b \\ \sqrt{1-\gamma}\, b^* & (1-\gamma)(1-a) \end{pmatrix}.$$

The population $(1-a)$ on $|1\rangle$ decays exponentially; the coherences b decay as $\sqrt{1-\gamma}$. This is the source of the $T_2 \le 2T_1$ relationship derived in Node 5.4: the coherence decay rate is at least half the population decay rate, because part of the dephasing comes from population decay itself.

The channel is non-unital: $\mathcal{E}(I) \ne I$ (the output is biased toward $|0\rangle\langle 0|$).

Stinespring dilation of amplitude damping

The amplitude damping channel can be realized as a unitary on the qubit plus an environment qubit, initially in $|0\rangle_E$:

$$V : |0\rangle_S \otimes |0\rangle_E \to |0\rangle_S \otimes |0\rangle_E,$$
$$V : |1\rangle_S \otimes |0\rangle_E \to \sqrt{1-\gamma}\, |1\rangle_S \otimes |0\rangle_E + \sqrt{\gamma}\, |0\rangle_S \otimes |1\rangle_E.$$

Tracing out the environment gives exactly the amplitude damping Kraus operators. The physical picture: the excited state $|1\rangle$ "leaks" into an entangled superposition where, with amplitude $\sqrt{\gamma}$, the system decays to ground and the environment absorbs a quantum of excitation. This is the quantum-mechanical content of spontaneous emission.

Measurement as a channel

A projective measurement with outcomes $\{P_i\}$ can be described as a channel whose Kraus operators are the projectors themselves (assuming outcomes are forgotten after measurement):

$$\mathcal{E}_{\text{meas}}(\rho) = \sum_i P_i \rho P_i.$$

This is a unital CPTP map that implements "dephasing in the measurement basis." If the outcome is retained as classical information, the channel becomes a *quantum-classical channel* $\rho \mapsto \sum_i \operatorname{Tr}(P_i\rho)|i\rangle\langle i|$, which destroys quantum coherence entirely and outputs a classical distribution.

Cross-Links

- **Module 5.2 (Reduced density operator):** The partial trace is a specific channel (tracing out subsystems). Every channel arises this way via Stinespring.
- **Module 5.4 (Decoherence mechanisms):** Dephasing and amplitude damping channels are the canonical quantum noise models, and are decoherence channels.
- **Module 7.4 (Von Neumann entropy):** Data-processing inequality $S(\mathcal{E}(\rho)) \geq S(\rho)$ for unital channels; mutual information is contractive under channels.
- **Module 8.2 (Lindblad equation):** The generator of a continuous-time semigroup of channels. Lindblad operators are the infinitesimal analogue of Kraus operators.
- **Module 4.2 (Separability):** Partial transpose is positive but not completely positive — it is not a channel, which is why PPT can detect entanglement.
- **Statistics:** Classical stochastic (Markov) channels are the commutative analogue: linear, completely positive (trivially), trace-preserving maps on probability distributions.

Structural Compression

Invariant: A quantum channel is a completely positive trace-preserving linear map. This is the unique class of transformations representing physically realizable operations on quantum states — neither smaller nor larger.

Mechanism: Three equivalent representations: Stinespring dilation (unitary on a larger system + partial trace), Kraus operator sum ($\sum_k K_k \rho K_k^\dagger$ with $\sum K_k^\dagger K_k = I$), and the Choi matrix via the Choi–Jamiołkowski isomorphism. Complete positivity is strictly stronger than positivity, forced by the existence of entangled ancillas.

Cross-System Echo: Stochastic matrices on probability distributions are the classical (commutative) analogue: linear, positive, trace-preserving. CPTP maps are the non-commutative generalization. Complete positivity — the strengthening of positivity to allow entangled inputs — is the specifically quantum ingredient.

Exercises

1. Show that the transpose map $T : \rho \mapsto \rho^T$ is positive but not completely positive by computing $(T \otimes I)|\Phi^+\rangle\langle\Phi^+|$ and finding its eigenvalues.
2. Verify that the Kraus operators of the depolarizing channel satisfy $\sum_k K_k^\dagger K_k = I$, and compute the action on the Bloch vector.
3. Derive the amplitude damping Kraus operators from a Stinespring dilation using a qubit environment.
4. Compute the Choi matrix for the depolarizing channel with $p = 1/2$. Verify that it is positive and satisfies the trace condition.

5. Show that the composition of two channels is a channel by constructing its Kraus representation from the individual Kraus operators.
6. Define a non-unital channel and discuss its fixed points. What is the fixed point of the amplitude damping channel?
7. Argue, structurally, why complete positivity (rather than mere positivity) is the correct mathematical requirement for physical channels. What would go wrong if we only required positivity?

Quantum Foundations · Module 7 — Quantum Information · Node 7.3
Concepts: quantum-channels, density-matrix, tensor-product, linear-operators
Prerequisites: 5.1, 5.2
Enables: 7.4, 8.2

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