Teleportation and Superdense Coding

Quantum Foundations · Module 7 — Quantum Information

Node 7.6

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Quantum Information **Node:** 7.6

Quantum teleportation (Bennett, Brassard, Crépeau, Jozsa, Peres, Wootters 1993) and **superdense coding** (Bennett–Wiesner 1992) are two protocols built on the same resource — a shared entangled Bell pair — that trade classical and quantum communication against each other in exactly reciprocal ways. Teleportation spends one Bell pair plus two classical bits to transmit one qubit. Superdense coding spends one Bell pair plus one qubit to transmit two classical bits. Together they establish entanglement as a *tradeable resource* that can be converted into classical or quantum communication and back, and they form the operational cornerstone of the theory of quantum information as a resource theory.

Beyond their foundational role, teleportation is the basic primitive for quantum networking, distributed quantum computation, and quantum state transfer over noisy channels. It is the closest real physical phenomenon to "beaming" — with the caveat that the original is destroyed in the process (required by no-cloning) and that no faster-than-light signaling is possible (classical information must accompany the entanglement).

Formal Statement

Teleportation protocol

Alice wants to transmit an unknown qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob. They share a Bell pair $|\Phi^+\rangle_{A'B} = (|00\rangle + |11\rangle)/\sqrt{2}$, with qubit A' held by Alice and B held by Bob. The protocol:

1. Alice performs a Bell-basis measurement on her two qubits (the unknown $|\psi\rangle$ and her half of the Bell pair). The measurement has four possible outcomes: $\Phi^+, \Phi^-, \Psi^+, \Psi^-$.
2. Alice sends the 2-bit classical outcome to Bob.
3. Depending on the outcome, Bob applies a Pauli correction to his half of the Bell pair:
   - $\Phi^+ \to I$ (no correction),
   - $\Phi^- \to Z$,
   - $\Psi^+ \to X$,
   - $\Psi^- \to XZ$ (or equivalently Y up to phase).

After the correction, Bob's qubit is in the state $|\psi\rangle$. Alice's original qubit, after the Bell measurement, is no longer in the state $|\psi\rangle$ — consistent with no-cloning.

Resource accounting: 1 qubit teleported = 1 Bell pair (1 "ebit") + 2 classical bits.

Superdense coding protocol

Alice wants to transmit two classical bits $(a, b) \in \{0, 1\}^2$ to Bob. They share a Bell pair $|\Phi^+\rangle_{AB}$. The protocol:

1. Alice applies one of four Pauli operators to her half of the Bell pair, depending on (a, b):

- $(0,0) \to I$
- $(0,1) \to X$
- $(1,0) \to Z$
- $(1,1) \to XZ$

2. Alice sends her half of the Bell pair to Bob via a quantum channel.
3. Bob performs a Bell-basis measurement on both qubits and recovers the 2-bit message.

The four possible operations map $|\Phi^+\rangle$ into the four Bell states $\{\Phi^+, \Phi^-, \Psi^+, \Psi^-\}$, which Bob can perfectly distinguish because they are orthogonal.

Resource accounting: 2 classical bits transmitted = 1 Bell pair + 1 qubit.

Reciprocity

The two protocols are exactly reciprocal in their resource conversions: - Teleportation: 2 cbits + 1 ebit $\to$ 1 qubit. - Superdense coding: 1 qubit + 1 ebit $\to$ 2 cbits.

A Bell pair (ebit) is the pivot, and the direction of resource flow depends on whether you are spending ebits to send classical information or quantum information.

Intuition

Teleportation's surprising feature is that Alice can transmit an *unknown* quantum state — one she cannot even describe, since describing it would require infinitely many classical bits (continuous amplitudes) — using only 2 classical bits. The "extra" information required to specify $|\psi\rangle$ is pre-packaged in the entanglement of the Bell pair. The Bell pair is a kind of "shared randomness with quantum flavor," and once Alice performs her Bell measurement, her measurement outcome is maximally random (each outcome has probability 1/4) regardless of $|\psi\rangle$. The specific random outcome, when combined with the Bell pair's pre-existing entanglement, picks out the correct unitary correction at Bob's end.

No faster-than-light signaling is possible. Before receiving Alice's 2 classical bits, Bob's qubit is the reduced density matrix $I/2$ — the maximally mixed state — which contains no information about $|\psi\rangle$ whatsoever. Only after Bob applies the correct Pauli correction (which requires knowing Alice's outcome) does his qubit become $|\psi\rangle$. The information is carried by the combination of the entanglement (already in place) and the classical message (traveling at most at light speed).

Superdense coding's surprise is that Alice can transmit *two* classical bits by sending only *one* qubit. Without shared entanglement, this would violate the Holevo bound. But with the Bell pair, Alice's qubit is never "alone" — it is part of a 4-dimensional entangled system, and the operations Alice applies to her half reveal themselves as distinct orthogonal states when measured jointly with Bob's half. The effective information capacity is $\log 4 = 2$ bits, matching what superdense coding achieves.

The reciprocity is structural: both protocols exploit the fact that the Bell basis is a 4-dimensional space in which any of the four distinct orthogonal states encodes 2 bits of classical information. Teleportation uses Alice's Bell measurement to "read out" which state the joint system is in (giving 2 classical bits); superdense coding uses Alice's Pauli operation to "write in" one of 4 Bell states (encoding 2 classical bits). The two are Fourier-dual in a sense.

Structural Role

- **Entanglement as a resource.** Teleportation and superdense coding establish that entanglement is not just a static correlation but a dynamical resource that can be converted into communication. This observation is the seed of quantum resource theories (Module 10.5), in which entanglement is quantified by how much it can do under restricted operations (LOCC, typically).

- **Classical vs quantum communication trade-offs.** The two protocols give the canonical answer to "how do classical bits and quantum bits trade against each other?" — through the intermediary of shared ebits. The full theory of such trade-offs (classical capacity, quantum capacity, entanglement-assisted capacity) is built on these primitives.
- **Quantum networking.** Teleportation is the basic primitive for sending quantum states over long distances. Quantum repeaters — the enabling technology for long-distance quantum communication — work by chaining multiple short-distance teleportation steps together, with entanglement distillation in between to combat noise.
- **Distributed quantum computation.** Gate teleportation generalizes state teleportation: a quantum gate can be "teleported" from one location to another using pre-shared entanglement and classical communication, enabling distributed quantum computation across separated processors.
- **Consistency with no-cloning.** Teleportation is not cloning: Alice's original qubit is destroyed by her Bell measurement, and only Bob's qubit retains $|\psi\rangle$. No-cloning would forbid any protocol that produced two copies; teleportation produces one copy and destroys the original, staying compatible.
- **Measurement-based quantum computation.** One-way quantum computation (Raussendorf–Briegel) is built on the idea that teleportation-like steps, repeated through a large entangled resource state, can implement arbitrary quantum computations. Teleportation is the local primitive; a graph state is the global resource.

Mathematical Development

Teleportation derivation

Alice holds qubits S (the unknown state) and A' (her half of the Bell pair); Bob holds qubit B (the other half). The total state is

$$|\psi\rangle_S \otimes |\Phi^+\rangle_{A'B} = (\alpha|0\rangle_S + \beta|1\rangle_S) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{A'B} + |11\rangle_{A'B}).$$

Expand and regroup in the Bell basis of qubits S and A':

$$\begin{aligned} &= \frac{1}{2}\Big[|\Phi^+\rangle_{SA'}(\alpha|0\rangle + \beta|1\rangle)_B + |\Phi^-\rangle_{SA'}(\alpha|0\rangle - \beta|1\rangle)_B \\ &\quad + |\Psi^+\rangle_{SA'}(\alpha|1\rangle + \beta|0\rangle)_B + |\Psi^-\rangle_{SA'}(\alpha|1\rangle - \beta|0\rangle)_B\Big]. \end{aligned}$$

The four terms correspond to the four possible Bell measurement outcomes, each with probability 1/4.

Alice measures qubits S and A' in the Bell basis. Suppose she obtains outcome Ψ^+. Bob's qubit is projected to $\alpha|1\rangle + \beta|0\rangle$, which equals $X|\psi\rangle$ — so Bob must apply X to recover $|\psi\rangle$.

The full outcome-to-correction map:

Alice's outcome	Bob's state	Correction
Φ^+	$\alpha\|\|0\rangle + \beta\|\|1\rangle$	I
Φ^-	$\alpha\|\|0\rangle - \beta\|\|1\rangle$	Z
Ψ^+	$\alpha\|\|1\rangle + \beta\|\|0\rangle$	X
Ψ^-	$\alpha\|\|1\rangle - \beta\|\|0\rangle$	XZ

After the appropriate correction, Bob's qubit is in state $|\psi\rangle$, regardless of the original amplitudes α, β.

Reduced state of Bob before classical communication

Before Bob receives Alice's outcome, his qubit's state is the reduced density matrix

$$\rho_B = \mathrm{Tr}_{SA'}\left(|\psi\rangle\langle\psi|_S \otimes |\Phi^+\rangle\langle\Phi^+|_{A'B}\right) = \frac{I}{2}.$$

This is independent of $|\psi\rangle$ — Bob has no access to the teleported state until Alice's 2 classical bits arrive. The protocol is consistent with no-signaling: classical communication is required to complete it.

Superdense coding derivation

Start with a Bell pair $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ shared between Alice (qubit A) and Bob (qubit B). Alice applies one of four Pauli operations to her qubit:

- $I \otimes I|\Phi^+\rangle = |\Phi^+\rangle$
- $Z \otimes I|\Phi^+\rangle = (|00\rangle - |11\rangle)/\sqrt{2} = |\Phi^-\rangle$
- $X \otimes I|\Phi^+\rangle = (|10\rangle + |01\rangle)/\sqrt{2} = |\Psi^+\rangle$
- $XZ \otimes I|\Phi^+\rangle = (|10\rangle - |01\rangle)/\sqrt{2} = |\Psi^-\rangle$ (up to an overall sign)

The four resulting Bell states are mutually orthogonal. Alice then sends qubit A to Bob, who now holds both qubits and performs a Bell measurement, recovering with certainty which of the four Bell states is present and hence which 2-bit message Alice encoded.

Resource accounting via coherent information

The coherent information $I_c(A\rangle B) = S(B) - S(AB)$ measures the quantum capacity of a channel. For a noiseless channel with prior ebit assistance, the coherent information of 1 qubit is $\log 2 = 1$, matching teleportation's rate. Entanglement-assisted classical capacity (Bennett–Shor–Smolin–Thapliyal 2002) gives $2 \log d$ for a noiseless channel, consistent with superdense coding.

These are special cases of the **resource calculus** of quantum Shannon theory, in which inequalities relate rates of qubit, cbit, and ebit consumption. Teleportation and superdense coding are the tightest extremal resource conversions in this calculus.

Worked Example

Teleporting $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ explicitly

Initial state (Alice holds qubits 1 and 2, Bob holds qubit 3):

$$|\psi\rangle_1|\Phi^+\rangle_{23} = (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Expand:

$$= \frac{1}{\sqrt{2}}\big(\alpha|0\rangle|00\rangle + \alpha|0\rangle|11\rangle + \beta|1\rangle|00\rangle + \beta|1\rangle|11\rangle\big).$$

Rewrite the joint state of qubits 1 and 2 in the Bell basis:

$$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Phi^-\rangle), \quad |11\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle - |\Phi^-\rangle),$$
$$|01\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle + |\Psi^-\rangle), \quad |10\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle - |\Psi^-\rangle).$$

Substituting and collecting terms (tedious but mechanical):

$$
\begin{aligned}
&= \frac{1}{2}\big[|\Phi^+\rangle_{12}(\alpha|0\rangle_3 + \beta|1\rangle_3) + |\Phi^-\rangle_{12}(\alpha|0\rangle_3 - \beta|1\rangle_3)\\
&\quad + |\Psi^+\rangle_{12}(\alpha|1\rangle_3 + \beta|0\rangle_3) + |\Psi^-\rangle_{12}(\alpha|1\rangle_3 - \beta|0\rangle_3)\big].
\end{aligned}
$$

Alice measures qubits 1 and 2 in the Bell basis. Each outcome has probability $1/4$. If the outcome is Φ^+, qubit 3 is already in $|\psi\rangle$; no correction needed. If it is Φ^-, Bob applies Z, turning $\alpha|0\rangle - \beta|1\rangle$ into $\alpha|0\rangle + \beta|1\rangle = |\psi\rangle$. Similarly for the other outcomes. In all four cases, after the classical communication and Pauli correction, Bob's qubit is $|\psi\rangle$.

Superdense coding, two-bit message "11"

Alice wants to send $(1, 1)$. She applies XZ to her half of $|\Phi^+\rangle$:

$$
(XZ \otimes I)|\Phi^+\rangle = (XZ \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) = |\Psi^-\rangle \text{ (up to sign).}
$$

Alice sends her qubit to Bob. Bob performs a Bell measurement and obtains Ψ^- with certainty, decoding "11." Total cost: 1 physical qubit transmitted + 1 Bell pair used = 2 bits of classical information delivered, at a rate of 2 cbits per qubit — double what any unassisted quantum channel could do, and matching the Holevo bound for 2-qubit systems.

Entanglement swapping

Combine teleportation with a second entangled pair to perform **entanglement swapping**: Alice holds qubits 1 and 2 (with 1 entangled with a distant qubit 0, and 2 entangled with Bob's qubit 3). Alice performs a Bell measurement on qubits 1 and 2. After classical communication and Bob's correction, qubits 0 and 3 — which have never interacted directly — become entangled. This is the primitive for quantum repeaters: long-distance entanglement is built up by swapping a chain of short-distance entangled pairs.

Cross-Links

- **Module 4.4 (Bell states and CHSH):** The Bell basis is the core ingredient in both protocols. Bell states are both the resource (Bell pair in teleportation) and the measurement basis.
- **Module 7.1 (Qubits):** The basic building blocks for the protocols.
- **Module 7.2 (No-cloning):** Teleportation is consistent with no-cloning because the original state is destroyed by Alice's Bell measurement.
- **Module 7.5 (Holevo bound):** Superdense coding saturates the Holevo bound for 2-qubit systems. It does not violate the bound because the relevant ensemble is on two qubits, not one.
- **Module 4.6 (Monogamy of entanglement):** Monogamy explains why teleportation uses up the Bell pair — the entanglement is "spent" in the Bell measurement and the pair becomes unentangled afterwards.
- **Module 10.5 (Quantum resource theories):** Teleportation and superdense coding are the foundational conversions in the resource theory of entanglement.
- **Statistics — shared randomness:** Classical shared randomness allows analogous (but much weaker) protocols; the quantum extension via shared entanglement gives exponentially greater power.

Structural Compression

Invariant: Shared entanglement is a tradeable resource that converts between classical and quantum communication. Teleportation: 1 ebit + 2 cbits → 1 qubit. Superdense coding: 1 ebit + 1 qubit → 2 cbits. The two protocols are reciprocal and exploit the 4-dimensional structure of the Bell basis.

Mechanism: Bell-basis measurement (teleportation) or Pauli encoding (superdense coding) on a pre-shared Bell pair, followed by classical communication (teleportation) or quantum transmission (superdense coding), followed by Pauli correction (teleportation) or Bell measurement (superdense coding).

Cross-System Echo: Shared randomness in classical communication allows rate-increasing protocols analogous to superdense coding (but much weaker, since shared random bits cannot transmit 2 cbits per 1 cbit of channel use). The quantum case is strictly stronger because Bell states have 4 orthogonal alternatives rather than 2, doubling the channel capacity with entanglement assistance.

Exercises

1. Derive the teleportation protocol by expanding $|\psi\rangle_S|\Phi^+\rangle_{A'B}$ in the Bell basis of SA'. Verify all four Pauli corrections.
2. Show explicitly that the reduced state of Bob's qubit before classical communication is $I/2$, independent of $|\psi\rangle$. Use this to argue that teleportation is consistent with no-signaling.
3. Verify that the four states $\{(I\otimes I)|\Phi^+\rangle, (Z\otimes I)|\Phi^+\rangle, (X\otimes I)|\Phi^+\rangle, (XZ\otimes I)|\Phi^+\rangle\}$ are mutually orthogonal and form a complete basis of two qubits.
4. Construct the entanglement swapping protocol: given Bell pairs $(0, 1)$ and $(2, 3)$, describe the measurement and corrections that entangle qubits 0 and 3.
5. Compute the success probability of teleportation if the shared Bell pair is imperfect, e.g., a Werner state $\rho = p|\Phi^+\rangle\langle\Phi^+| + (1-p)I/4$. At what p does teleportation fidelity drop below classical?
6. Describe gate teleportation: how to apply a known single-qubit gate U to a distant qubit using pre-shared entanglement and classical communication.
7. Argue, structurally, why teleportation and superdense coding are reciprocal in their resource consumption. What role does the 4-dimensional Bell basis play in both directions?

Quantum Foundations · Module 7 — Quantum Information · Node 7.6
Concepts: entanglement, tensor-product, unitary-evolution, information-theory
Prerequisites: 7.1, 7.2, 4.4

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