Markovian vs Non-Markovian Dynamics

Quantum Foundations · Module 8 — Open Systems

Node 8.3

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Open Quantum Systems **Node:** 8.3

The Lindblad equation (Node 8.2) describes **Markovian** dynamics — open-system evolution in which the environment has no memory, and the future state depends only on the present. This is the cleanest and most widely used class of open-system models, but it is an approximation. Real environments are finite, structured, and often slow compared to the system they couple to. When these conditions break down, memory effects appear: the environment "remembers" what the system did in the past and feeds that information back later, producing **non-Markovian** dynamics with features such as population revivals, non-monotonic coherence decay, and information backflow.

This node makes the Markovian/non-Markovian distinction mathematically precise, introduces the standard measures of non-Markovianity (BLP and RHP), and shows through canonical examples (the damped Jaynes–Cummings model) how both regimes arise from the same microscopic physics at different coupling strengths. It also situates the distinction in the context of real hardware: modern quantum devices frequently exhibit non-Markovian noise, and characterizing it is essential for accurate error modeling and mitigation.

Formal Definition

Dynamical maps and divisibility

Given a family of CPTP maps $\{\Lambda_{t,0}\}_{t\geq 0}$ describing the evolution from time 0 to time t, the dynamics is called **CP-divisible** if for every pair $0 \leq s \leq t$, there exists a CPTP map $V_{t,s}$ such that

$$\Lambda_{t,0} = V_{t,s} \circ \Lambda_{s,0}.$$

That is, the evolution from 0 to t can be broken into an evolution from 0 to s followed by a *valid quantum channel* from s to t.

Markovian and non-Markovian

A dynamics is **Markovian** (in the sense of Rivas–Huelga–Plenio, 2010) if it is CP-divisible. Equivalently, the dynamical map satisfies a time-local master equation of Lindblad form, possibly with time-dependent rates that remain nonnegative throughout the evolution:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H(t), \rho] + \sum_k \gamma_k(t)\mathcal{D}[L_k(t)](\rho), \qquad \gamma_k(t) \geq 0 \quad \forall\, t.$$

If the rates $\gamma_k(t)$ are constant, the dynamics is also *time-homogeneous* and satisfies the full semigroup property $\Lambda_{t+s} = \Lambda_t \circ \Lambda_s$.

A dynamics is **non-Markovian** if it is not CP-divisible. This manifests as either: - A time-local master equation with some rate $\gamma_k(t) < 0$ at some time (while the full map $\Lambda_{t,0}$ remains CPTP), or - A non-time-local integro-differential master equation with memory kernels (the Nakajima–Zwanzig form).

BLP measure

The **Breuer–Laine–Piilo (BLP) measure** (2009) quantifies non-Markovianity via information backflow. Define the trace distance between two states as $D(\rho_1, \rho_2) = \frac{1}{2}\|\rho_1 - \rho_2\|_1$. Under any CPTP map, trace distance is contractive:

$$D(\Lambda(\rho_1), \Lambda(\rho_2)) \leq D(\rho_1, \rho_2).$$

For Markovian dynamics, the trace distance between any pair of input states is monotonically non-increasing in time. For non-Markovian dynamics, the trace distance can *increase* during certain intervals — this is information backflow from the environment to the system. The BLP measure is

$$\mathcal{N}_{\text{BLP}} = \max_{\rho_1, \rho_2} \int_{\sigma > 0} \sigma(t)\, dt, \qquad \sigma(t) = \frac{d}{dt} D(\Lambda_{t,0}(\rho_1), \Lambda_{t,0}(\rho_2)),$$

where the integral is over time intervals in which the distance grows.

RHP measure

The **Rivas–Huelga–Plenio (RHP) measure** (2010) characterizes non-Markovianity via CP-divisibility directly. A map is CP-divisible iff $V_{t+dt,t}$ is CPTP for every t, which can be tested via the Choi matrix of $V_{t+dt,t}$. The RHP measure integrates the magnitude of non-CP contributions:

$$\mathcal{N}_{\text{RHP}} = \int_0^\infty g(t)\, dt,$$

where $g(t)$ measures the instantaneous deviation of $V_{t+dt,t}$ from CPTP.

These measures are not equivalent in general: CP-divisibility implies trace-distance monotonicity, but the converse does not hold — there exist non-CP-divisible dynamics whose BLP measure vanishes. RHP is stricter than BLP.

Intuition

The Markovian picture is: the environment is so large and its correlations decay so fast that any perturbation the system induces in it is "washed away" before it can come back to affect the system. The environment is like an ocean — throw a pebble in, and the ripples disperse. The system evolves as if the environment were always in its equilibrium state, and each infinitesimal time step is statistically independent of the past.

The non-Markovian picture is: the environment is finite, structured, or slow enough that it "remembers" what the system did. A perturbation injected into the environment doesn't disperse — it propagates, reflects off environmental structure, and comes back to affect the system again. The environment is more like a resonant cavity than an ocean. The system's future depends not just on its current state but on the history of its interactions with the environment.

Practical consequences:

- **Population revivals.** Instead of exponential decay $e^{-\gamma t}$, non-Markovian populations can oscillate or even fully return to their initial values at certain times.
- **Non-monotonic coherence decay.** Coherences may decay, then partially recover, then decay again.

- **Information backflow.** The distinguishability of states (trace distance) increases at certain times, meaning information that leaked to the environment is coming back.
- **Enhanced quantum control.** Non-Markovian dynamics, if characterized and exploited, can be used to recover information via engineered echo sequences or feedback protocols.

The canonical physical example is a two-level atom strongly coupled to a single-mode cavity (the Jaynes–Cummings model): in the weak-coupling limit, the cavity acts as a Markovian environment and the atom decays exponentially; in the strong-coupling limit, the atom and cavity exchange an excitation coherently, producing vacuum Rabi oscillations — a textbook non-Markovian revival.

The distinction matters for real hardware. Superconducting qubits often couple to a few nearby two-level systems (two-level defects, spurious resonators) rather than to a smooth environment, and the resulting noise is non-Markovian. Characterizing this non-Markovianity is essential for high-fidelity gate operations and error mitigation.

Structural Role

- **Classifying open-system dynamics.** The Markovian/non-Markovian distinction is the fundamental classification of open-system dynamics, analogous to the classical distinction between Markov processes and processes with memory.
- **Range of validity of the Lindblad equation.** The Lindblad equation is valid only in the Markovian regime. For non-Markovian dynamics, one needs either a time-local master equation with possibly negative rates (yielding a CPTP dynamical map but not a semigroup), a Nakajima–Zwanzig memory kernel equation (exact but typically intractable), or an enlargement of the system Hilbert space to explicitly include relevant environmental degrees of freedom.
- **Noise mitigation strategies.** Markovian noise is fought with repeated error correction or dynamical decoupling optimized for white-noise spectra. Non-Markovian noise can be mitigated by tailored pulse sequences that exploit the environment's memory structure (e.g., CPMG sequences that refocus low-frequency environmental fluctuations).
- **Resource-theoretic perspective.** Non-Markovianity can be treated as a *resource* in the sense that it allows information backflow — this backflow can be exploited for tasks like state preparation, quantum control, and quantum sensing. The resource theory of non-Markovianity studies what operations preserve or deplete this resource.
- **Bridge to exact dynamics.** Non-Markovian master equations are intermediate between fully exact system+environment dynamics and the Lindblad approximation. They retain some features of the exact dynamics (memory effects, revivals) while still operating on the reduced density matrix of the system.

Mathematical Development

Time-local master equation with time-dependent rates

A general time-local master equation can be written as

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H(t),\rho] + \sum_k \gamma_k(t)\left(L_k(t)\rho L_k(t)^\dagger - \frac{1}{2}\{L_k(t)^\dagger L_k(t),\rho\}\right).$$

Markovian case: $\gamma_k(t) \geq 0$ for all t and all k. The dynamical map is CP-divisible.

Non-Markovian case: some $\gamma_k(t) < 0$ at some times. Although the individual infinitesimal generator is not Lindblad-like, the time-ordered exponential can still yield a CPTP map $\Lambda_{t,0}$ at each time. However, the intermediate maps $V_{t,s}$ for $0 < s < t$ may fail to be CP, violating divisibility. The negative rates are the mathematical manifestation of information backflow.

Nakajima–Zwanzig memory kernel

The exact reduced dynamics of a system coupled to an environment can be derived via the Nakajima–Zwanzig projection technique, giving an integro-differential equation

$$\frac{d\rho_S(t)}{dt} = -\frac{i}{\hbar}[H_S, \rho_S(t)] + \int_0^t \mathcal{K}(t-s)\rho_S(s)\, ds,$$

where $\mathcal{K}(t-s)$ is a memory kernel encoding the environment's influence on the system's past. The Markov approximation is recovered when the kernel is sharply peaked at $t = s$, i.e., $\mathcal{K}(t-s) \approx \delta(t-s)\mathcal{L}_M$ for some Lindbladian $\mathcal{L}_M$. When the kernel has finite width, memory effects appear.

Damped Jaynes–Cummings model

Consider a two-level atom with transition frequency ω_0 coupled to a single cavity mode of frequency ω_c and decay rate κ (the cavity leaks photons to an external reservoir at rate κ). The atom-cavity coupling has strength g.

In the single-excitation sector, the exact reduced dynamics of the atom's excited state amplitude $c_e(t)$ satisfies an integro-differential equation whose solution depends on the ratio g/κ:

- **Weak coupling $g \ll \kappa$:** The cavity decays faster than the atom-cavity exchange, and the cavity acts as a Markovian environment for the atom. The atomic excited state decays exponentially with effective rate $\Gamma = 4g^2/\kappa$ (Purcell-enhanced spontaneous emission).
- **Strong coupling $g \gg \kappa$:** The atom-cavity exchange is faster than cavity decay. An excitation oscillates coherently between atom and cavity (vacuum Rabi oscillations) with frequency $2g$, decaying slowly on the scale κ. The atomic population exhibits revivals, and the dynamics is strongly non-Markovian.

The crossover at $g \sim \kappa$ is continuous, and the BLP/RHP measures capture the increasing non-Markovianity smoothly through the transition.

BLP example

Take a qubit with the non-Markovian amplitude-damping dynamics from the strong-coupling Jaynes–Cummings model. Start with two initial states $|+\rangle$ and $|-\rangle$. Their trace distance at time t is $|\cos(gt)|e^{-\kappa t/2}$ (schematically) — oscillating between small and large values due to the Rabi oscillation.

The time derivative $\sigma(t) = dD/dt$ is negative during coherence decay intervals but positive during revivals. Integrating over the revival intervals gives a nonzero BLP measure, quantifying the non-Markovianity.

Beyond Lindblad: positivity-preserving vs CP-divisible

An important subtlety: positivity-preserving dynamics is weaker than CP-divisible dynamics. There are dynamical maps that preserve positivity (so any single input state evolves to a valid density matrix) but fail CP-divisibility, because they fail to preserve positivity when tensored with an ancilla (analogous to the transpose map's failure of CP from Node 7.3). These dynamics are physically realizable but require accounting for non-trivial correlations between system and environment that violate CP-divisibility.

Worked Example

Two regimes of spontaneous emission in a Lorentzian bath

Consider a two-level atom in a photonic bath with a Lorentzian spectral density peaked at the atomic transition frequency ω_0:

$$J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{(\omega - \omega_0)^2 + \lambda^2},$$

where λ sets the width of the Lorentzian and γ_0 sets the coupling strength.

Regime 1: $\lambda \gg \gamma_0$ (broad bath, weak coupling). The bath correlation function decays on timescale $1/\lambda$, much faster than the system timescale $1/\gamma_0$. Markov approximation applies; the atom decays exponentially with rate γ_0. This is the Wigner–Weisskopf theory of spontaneous emission.

Regime 2: $\lambda \ll \gamma_0$ (narrow bath, strong coupling). The bath is essentially a single resonant mode with slow leakage. An excitation oscillates between the atom and the narrow-bath peak, with partial revivals of the atomic excited-state population. The dynamics is strongly non-Markovian, and the exact reduced dynamics (solvable in this model) shows the atomic amplitude obeying

$$c_e(t) = e^{-\lambda t/2} \left[\cosh\left(\frac{\Omega t}{2}\right) + \frac{\lambda}{\Omega} \sinh\left(\frac{\Omega t}{2}\right) \right], \qquad \Omega = \sqrt{\lambda^2 - 2\gamma_0 \lambda}.$$

When $2\gamma_0 > \lambda$, Ω becomes imaginary and the population oscillates with revivals — the non-Markovian regime.

Real hardware: superconducting qubits with TLS

Superconducting transmon qubits couple to amorphous two-level system (TLS) defects in oxide layers and interfaces. A single TLS with frequency close to the qubit's transition acts as a structured, non-Markovian bath. Instead of decaying exponentially, the qubit can exchange excitations with the TLS, producing characteristic "avoided crossing" patterns in spectroscopy and population revivals in time-domain experiments. Modern device characterization must account for this non-Markovian structure to accurately predict gate fidelities.

Environment engineering for error correction

Because non-Markovian noise has memory, it can sometimes be turned into an advantage. Dynamical decoupling sequences such as CPMG refocus slow environmental fluctuations by rapid pulse echoes, effectively suppressing the non-Markovian component of the noise and making the effective dynamics closer to Markovian white noise — which is easier to protect against with standard error correction. The success of dynamical decoupling depends critically on the non-Markovian spectrum being narrow compared to the pulse repetition rate.

Cross-Links

- **Module 8.2 (Lindblad equation):** Describes exactly the Markovian regime. Non-Markovian dynamics requires generalization beyond GKSL.
- **Module 5.3 (System-environment coupling):** The Born–Markov approximation is the microscopic justification for Markovianity; its failure is the origin of non-Markovian dynamics.
- **Module 8.4 (Quantum trajectories):** Markovian trajectories have discrete jumps; non-Markovian trajectories involve correlations across jumps.
- **Module 5.6 (Decoherence timescales):** Non-Markovian baths can produce deviations from simple exponential T_1, T_2 decay, requiring more sophisticated characterization.
- **Module 8.6 (Error correction):** Error correction thresholds depend on the noise structure; non-Markovian noise often has correlated errors that degrade standard threshold estimates.
- **Statistics — Markov chains:** Classical Markov processes are the structural analogue of Markovian quantum dynamics. Non-Markovian quantum dynamics echoes classical processes with memory (e.g., fractional Brownian motion, autoregressive processes).

Structural Compression

Invariant: A quantum dynamical map is Markovian iff it is CP-divisible, iff the associated time-local master equation has nonnegative rates at all times. Non-Markovianity manifests as information backflow — increase of trace distance between distinct states during evolution — and is quantified by the BLP or RHP measures.

Mechanism: Markovianity requires environment correlations to decay faster than system dynamics; this justifies the Born–Markov approximation leading to the Lindblad equation. Non-Markovianity arises when the environment is finite, structured, or slow, causing memory effects encoded in a non-trivial Nakajima–Zwanzig memory kernel or time-dependent negative rates.

Cross-System Echo: Classical Markov chains have the memoryless property; classical processes with memory require higher-order state spaces or explicit memory kernels. The quantum distinction maps onto the classical one with one complication: CP-divisibility (the quantum Markov condition) is stronger than mere positivity-divisibility, reflecting the need to handle entangled ancilla systems.

Exercises

1. Show that the Lindblad equation with constant rates is CP-divisible, and identify the intermediate maps $V_{t,s}$.
2. Construct a time-local master equation with $\gamma(t) < 0$ during some interval, and show that although the rate is negative, the dynamical map can still remain CPTP.
3. Solve the Jaynes–Cummings model in the single-excitation sector exactly, and identify the crossover from Markovian to non-Markovian dynamics as a function of g/κ.
4. Compute the BLP measure for the non-Markovian amplitude damping model with Lorentzian spectral density in the strong-coupling regime.
5. Discuss the relationship between BLP and RHP measures: give an example (or describe one from the literature) of a dynamics that is non-CP-divisible but has vanishing BLP measure.
6. Explain how dynamical decoupling sequences like CPMG exploit the non-Markovian structure of a bath to suppress dephasing. What conditions on the bath spectrum make CPMG effective?
7. Argue, structurally, why CP-divisibility is the correct quantum notion of Markovianity, rather than the weaker condition of positivity-divisibility or the stronger condition of the full semigroup property.

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Concepts: markov-chains, quantum-channels, quantum-noise
Prerequisites: 8.2
Enables: 8.4

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