Quantum Trajectories

Quantum Foundations · Module 8 — Open Systems

Node 8.4

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Open Quantum Systems **Node:** 8.4

The Lindblad equation describes the *average* evolution of an open system — the density operator obtained by averaging over all possible histories of the environment. But a single run of a physical experiment with a single quantum system does not produce the average; it produces one specific random history. A **quantum trajectory** is a stochastic unraveling of the Lindblad equation into individual sample paths, each of which is a possible evolution of the system conditioned on a specific measurement record in the environment. Averaging over many trajectories recovers the deterministic Lindblad dynamics.

Quantum trajectories serve three distinct purposes. Numerically, they provide a Monte Carlo method — the **Monte Carlo wavefunction (MCWF)** or **quantum jump method** — for simulating large open systems by evolving pure states and taking statistical averages, which is often vastly cheaper than evolving the full density matrix. Conceptually, they clarify what a measurement "looks like" from the system's perspective: a continuous weak interaction with an environment that is being monitored, producing a stochastic trajectory of the conditional state. Experimentally, they describe exactly the state-of-the-art in continuous monitoring and feedback control of quantum systems, which is central to quantum error correction, quantum sensing, and cavity QED.

Formal Definition

Unraveling a Lindblad equation

Given a Lindblad master equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \sum_k \mathcal{D}[L_k](\rho),$$

an **unraveling** is a stochastic process for a pure state $|\psi(t)\rangle$ whose ensemble average reproduces $\rho(t)$:

$$\rho(t) = \mathbb{E}[|\psi(t)\rangle\langle\psi(t)|].$$

There are infinitely many possible unravelings, each corresponding to a different choice of how the environment is monitored. The two most important classes are **jump unravelings** (corresponding to photon counting) and **diffusive unravelings** (corresponding to homodyne detection).

Jump unraveling (Monte Carlo wavefunction)

In the jump picture, the state evolves deterministically under a non-Hermitian effective Hamiltonian

$$H_{\text{eff}} = H - \frac{i\hbar}{2}\sum_k L_k^\dagger L_k,$$

punctuated by discrete **quantum jumps** at random times. The jumps occur with rate

$$p_k(t)\,dt = \langle\psi(t)|L_k^\dagger L_k|\psi(t)\rangle\,dt,$$

and when a jump of type k occurs, the state is updated via

$$|\psi(t+dt)\rangle = \frac{L_k|\psi(t)\rangle}{\|L_k|\psi(t)\rangle\|}.$$

Between jumps, the deterministic non-Hermitian evolution

$$|\tilde{\psi}(t+dt)\rangle = \big(I - \frac{i}{\hbar}H_{\text{eff}}\,dt\big)|\psi(t)\rangle$$

followed by normalization gives the conditional state (normalized to account for the probability of no jump). The stochastic equation is

$$d|\psi\rangle = \Big(-\frac{i}{\hbar}H_{\text{eff}} + \frac{1}{2}\sum_k\langle L_k^\dagger L_k\rangle\Big)|\psi\rangle\,dt + \sum_k\Big(\frac{L_k}{\sqrt{\langle L_k^\dagger L_k\rangle}} - I\Big)|\psi\rangle\,dN_k,$$

where dN_k are Poisson increments with $\mathbb{E}[dN_k] = \langle L_k^\dagger L_k\rangle\,dt$ and $dN_k dN_l = \delta_{kl}dN_k$.

Diffusive unraveling (homodyne detection)

In the diffusive picture, the state evolves continuously but stochastically, driven by Wiener noise:

$$d|\psi\rangle = -\frac{i}{\hbar}H|\psi\rangle\,dt + \sum_k\Big(L_k - \frac{1}{2}\langle L_k + L_k^\dagger\rangle\Big)|\psi\rangle\,dW_k - \frac{1}{2}\sum_k\Big(L_k^\dagger L_k - \langle L_k + L_k^\dagger\rangle L_k + \frac{1}{4}\langle L_k + L_k^\dagger\rangle^2\Big)|\psi\rangle\,dt,$$

where dW_k are independent Wiener increments with $\mathbb{E}[dW_k] = 0$ and $dW_k dW_l = \delta_{kl}dt$. This corresponds to continuous monitoring of the environment quadratures (homodyne or heterodyne detection in quantum optics).

Equivalence on average

For both unravelings, taking the ensemble average $\rho(t) = \mathbb{E}[|\psi(t)\rangle\langle\psi(t)|]$ reproduces the Lindblad equation. The individual trajectories are different in the two pictures, but their ensemble average is invariant under the choice of unraveling.

Intuition

The master equation tells you the *ensemble average* of many experimental runs. But a single run produces a single specific history — one sample path. The quantum trajectory picture says: imagine that the environment is continuously monitored by a gedanken-experimenter, who records every photon emitted, every phase shift, every measurement outcome. Conditional on this full record, the system's state is always a pure state (the environment, being fully measured, cannot be entangled with the system). This pure conditional state evolves stochastically, driven by the randomness of the measurement outcomes.

When you don't monitor the environment (or you average over all possible monitoring outcomes), you recover the Lindblad density matrix. The density matrix is the ensemble average over all possible measurement histories, each of which corresponds to a single trajectory.

The "jump" vs "diffusive" distinction reflects the *type* of measurement performed on the environment:

- **Jump unraveling = photon counting.** The environment is monitored with detectors that click discretely when a photon is emitted. Between clicks, the system evolves deterministically (but non-Hermitianly, because the *absence* of a click is itself information that updates the state). At each click, the state jumps discontinuously. This is the natural description for photon emission, spontaneous decay, and discrete detection events.
- **Diffusive unraveling = homodyne detection.** The environment is monitored with a local oscillator that measures quadrature amplitudes continuously. The system state performs a continuous stochastic walk driven by white-noise measurement records. This is the natural description for continuous weak measurement and for many solid-state qubit readout schemes.

Both pictures describe the same underlying physics; they correspond to different detector designs. The system's density matrix, averaged over the unknown outcomes of these detectors, obeys the same Lindblad equation regardless of which detector is imagined. Trajectories are an interpretive and computational tool, not a new physical theory.

The Monte Carlo wavefunction method exploits this for numerical efficiency. A density matrix on N states has N^2 entries; a pure state has only N. Evolving a pure state stochastically and averaging over many realizations is often much cheaper than evolving the N^2-dimensional density matrix, especially when the system is large. This is the standard workhorse for simulating open quantum systems in quantum optics, condensed matter, and quantum chemistry.

Structural Role

- **Numerical methods.** The Monte Carlo wavefunction method (Dalibard–Castin–Mølmer, 1992) is one of the standard techniques for simulating open quantum systems. It reduces the computational cost from $O(N^2)$ per time step (full density matrix) to $O(N)$ per trajectory, at the cost of needing many trajectories for statistical convergence.
- **Continuous measurement theory.** Quantum trajectories are the mathematical foundation of continuous measurement, which underlies modern quantum sensing (gravitational wave detectors, atomic clocks) and quantum feedback control.
- **Conditional dynamics in quantum error correction.** Real-time error correction schemes like cat codes, GKP codes, and stabilizer codes with continuous monitoring use the conditional quantum state to decide when and how to apply corrections. The theory of quantum trajectories is the language in which these schemes are formulated.
- **Conceptual picture of measurement.** Quantum trajectories give a concrete picture of what happens during a measurement: the system's state evolves stochastically under the influence of measurement backaction, collapsing gradually (diffusive) or suddenly (jump) depending on the measurement type. This is the "dynamical collapse" picture that dissolves many apparent paradoxes of measurement.
- **Link to stochastic differential equations.** The stochastic Schrödinger equation is a quantum generalization of classical stochastic differential equations (Itô and Stratonovich calculi), and it inherits

the same mathematical structure and pitfalls. Quantum trajectories provide a rich testing ground for stochastic analysis.

Mathematical Development

Monte Carlo wavefunction algorithm

Given an initial pure state $|\psi(0)\rangle$, the MCWF algorithm proceeds in discrete time steps dt:

1. Compute the total jump probability $p(t)\,dt = \sum_k \langle L_k^\dagger L_k \rangle\, dt$.
2. Draw a uniform random number $r \in [0,1]$.
3. **If** $r < p(t)\,dt$: a jump occurs.
   - Choose which jump via weights p_k/p.
   - Apply $|\psi\rangle \to L_k|\psi\rangle/\|L_k|\psi\rangle\|$.
4. **Else:** no jump.
   - Apply the non-Hermitian evolution $|\psi\rangle \to |\tilde{\psi}\rangle = (I - iH_{\text{eff}}dt/\hbar)|\psi\rangle$ and renormalize: $|\psi\rangle = |\tilde{\psi}\rangle/\|\tilde{\psi}\|$.
5. Repeat.

Averaging over many trajectories gives $\rho(t)$ and all its statistical properties.

Proof of equivalence with Lindblad

To see that the ensemble average of the MCWF algorithm reproduces the Lindblad equation, compute

$$\rho(t+dt) = \mathbb{E}[|\psi(t+dt)\rangle\langle\psi(t+dt)|]$$

by conditioning on whether a jump occurred:

$$= (1 - p\,dt)\frac{(I - iH_{\text{eff}}dt/\hbar)|\psi\rangle\langle\psi|(I + iH_{\text{eff}}^\dagger dt/\hbar)}{1 - p\,dt} + \sum_k p_k\,dt \frac{L_k|\psi\rangle\langle\psi|L_k^\dagger}{\langle L_k^\dagger L_k\rangle}.$$

The normalization factors cancel, and expanding to first order in dt yields

$$\rho(t+dt) = \rho(t) + dt\left(-\frac{i}{\hbar}[H,\rho] + \sum_k L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}\right) + O(dt^2),$$

which is exactly the Lindblad equation. So the trajectory average reproduces the deterministic evolution.

Stochastic master equation

When the environment is partially monitored (say, with detection efficiency $\eta < 1$), the conditional state is neither a pure state (because of the unmonitored part of the environment) nor the full ensemble average (because the monitored part has been measured). It satisfies a **stochastic master equation (SME)** that interpolates between the Lindblad equation (at $\eta = 0$) and the pure-state unraveling (at $\eta = 1$).

For homodyne detection with efficiency η:

$$d\rho_c = -\frac{i}{\hbar}[H,\rho_c]dt + \mathcal{D}[L]\rho_c\,dt + \sqrt{\eta}\,\mathcal{H}[L]\rho_c\,dW,$$

where $\mathcal{H}[L]\rho = L\rho + \rho L^\dagger - \text{Tr}((L + L^\dagger)\rho)\rho$ is the "measurement innovation" superoperator. Averaging over the noise dW gives the unconditioned Lindblad equation.

Itô vs Stratonovich

The diffusive unraveling is naturally written in Itô form, where the noise increment is “non-anticipating” (does not know the future). Transforming to Stratonovich form (which uses the symmetric midpoint rule) adds extra drift terms, but the two are equivalent descriptions of the same process. Itô is usually preferred for computation; Stratonovich is sometimes more natural for physical derivations. Both give the same ensemble average.

Worked Example

Photon counting on a damped cavity

Consider a cavity mode with $H = \omega a^\dagger a$ and damping Lindblad operator $L = \sqrt{\kappa} a$. Start in a Fock state $|n\rangle$. The jump operator a lowers the photon number by 1 when a jump occurs; the non-Hermitian effective Hamiltonian is $H_{\text{eff}} = \omega a^\dagger a - i\hbar\kappa a^\dagger a/2$.

Between jumps, the state $|n\rangle$ evolves as

$$e^{-iH_{\text{eff}}t/\hbar}|n\rangle = e^{-i\omega nt-\kappa nt/2}|n\rangle,$$

with norm decaying at rate κn (the total jump rate for an n-photon state). The probability of no jump up to time t is $e^{-\kappa nt}$, so the expected time to the first jump is $1/(\kappa n)$.

A single trajectory looks like: start in $|n\rangle$, wait a random time $\tau_1 \sim \text{Exponential}(\kappa n)$, then jump to $|n-1\rangle$, wait another random time $\tau_2 \sim \text{Exponential}(\kappa(n-1))$, then jump to $|n-2\rangle$, and so on until reaching $|0\rangle$.

The ensemble average of these trajectories gives the density matrix $\rho(t) = \sum_m p_m(t)|m\rangle\langle m|$, where $p_m(t)$ is the probability of having m photons at time t. The moments match the Lindblad prediction exactly: $\langle a^\dagger a\rangle(t) = ne^{-\kappa t}$ (exponential decay of mean photon number).

Homodyne detection of a qubit

Consider a qubit with dephasing via $L = \sqrt{\gamma_\phi/2}Z$. The diffusive unraveling gives

$$d|\psi\rangle_c = -\frac{\gamma_\phi}{2}(Z-\langle Z\rangle)^2|\psi\rangle_c\,dt + \sqrt{\gamma_\phi/2}(Z-\langle Z\rangle)|\psi\rangle_c\,dW.$$

Starting from $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, the trajectory evolves stochastically. Over time, individual trajectories collapse toward either $|0\rangle$ (with $Z = +1$) or $|1\rangle$ (with $Z = -1$), chosen randomly with equal probability. The average of the trajectories gives $\rho(t) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ after long times — the same as the Lindblad prediction for the ensemble-averaged density matrix.

The *conditional* state, however, is always a pure state on the Z-eigenvector axis, gradually collapsing to an eigenstate as measurement information accumulates. This is the diffusive picture of measurement: continuous acquisition of information drives the state toward a measurement eigenstate via a stochastic process.

Numerical simulation of spontaneous emission

Simulating spontaneous emission from a two-level atom via MCWF: the effective Hamiltonian is $H_{\text{eff}} = -i\hbar\gamma\sigma_+\sigma_-/2$ (taking $H = 0$ in the rotating frame). The jump operator is $L = \sqrt{\gamma}\sigma_-$.

Start in $|1\rangle$. The non-Hermitian evolution gives $|\tilde{\psi}(t)\rangle = e^{-\gamma t/2}|1\rangle$, with norm squared $e^{-\gamma t}$, which is exactly the probability of no jump up to time t. At a random time (exponentially distributed with rate γ), the jump $|1\rangle \to |0\rangle$ occurs, and the state remains in $|0\rangle$ thereafter.

Each trajectory is a single step function: $|1\rangle$ until some random jump time, then $|0\rangle$. Averaging over many such trajectories gives the smooth exponential decay $p_1(t) = e^{-\gamma t}$. A single experiment produces one step function, not the ensemble curve — which is exactly what is observed in single-atom fluorescence experiments with time-resolved detection.

Cross-Links

- **Module 3.5 (Measurement theory):** Quantum trajectories formalize continuous weak measurement. Each trajectory is conditional on a specific measurement record.
- **Module 8.2 (Lindblad equation):** Ensemble averages of trajectories reproduce the Lindblad equation. Trajectories and the master equation are two views of the same dynamics.
- **Module 8.6 (Error correction):** Continuous-monitoring error correction schemes use conditional quantum states computed via trajectory methods to decide on corrective operations.
- **Module 5.4 (Decoherence mechanisms):** Amplitude damping and dephasing channels are the integrated forms of Lindblad dynamics whose trajectories are direct photon-counting or homodyne-detection processes.
- **Statistics — Stochastic processes:** Classical Poisson processes (jump unraveling) and Wiener processes (diffusive unraveling) are the mathematical backbone of trajectory theory.
- **VQA:** Stochastic gradient descent in optimization has structural parallels to diffusive unravelings — a deterministic flow with stochastic fluctuations whose ensemble average gives the "true" dynamics.

Structural Compression

Invariant: Any Lindblad master equation admits stochastic unravelings into pure-state trajectories whose ensemble average reproduces the deterministic evolution. The unravelings correspond to different choices of how the environment is monitored, and there are infinitely many valid unravelings consistent with the same master equation.

Mechanism: Monte Carlo wavefunction (jump unraveling, corresponding to photon counting) or stochastic Schrödinger equation (diffusive unraveling, corresponding to homodyne detection). In both cases, a pure state evolves under a non-Hermitian effective Hamiltonian between stochastic events (jumps or Wiener noise), and the ensemble average over trajectories reproduces the density-matrix evolution.

Cross-System Echo: Sample paths vs ensemble average in classical stochastic processes: a Brownian motion has individual trajectories that are continuous but rough, while the ensemble-averaged distribution is smooth and obeys a deterministic PDE (the heat equation). Quantum trajectories generalize this to the non-commutative setting, with pure states playing the role of sample points and density matrices playing the role of probability distributions.

Exercises

1. Show explicitly that the Monte Carlo wavefunction algorithm's ensemble average reproduces the Lindblad equation to first order in dt.
2. Simulate (on paper) two trajectories of a damped two-level atom in MCWF starting from $|1\rangle$. What are the possible trajectories and their probabilities?
3. Derive the diffusive stochastic Schrödinger equation from homodyne detection of a cavity decay, starting from the projection of the environment field onto homodyne outcomes.
4. For a qubit with dephasing $L = \sqrt{\gamma} Z$, compute the trajectory for a diffusive unraveling starting from $|+\rangle$. Show that long-time trajectories localize on Z eigenstates.
5. Compute the expected time to the first jump in a photon-counting simulation of spontaneous emission from the state $|n\rangle$.
6. Explain how different unravelings (jump vs diffusive) can both correspond to the same density-matrix evolution. What does this ambiguity say about the physical reality of the conditional state?

7. Argue, structurally, why the MCWF method is numerically efficient for large Hilbert spaces. For what kinds of systems does this method provide the largest computational advantage over direct density-matrix evolution?

Quantum Foundations · Module 8 — Open Systems · Node 8.4
Concepts: monte-carlo-methods, quantum-channels, density-matrix, simulation
Prerequisites: 8.2, 3.5
Enables: 8.5

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