Decoherence-Free Subspaces

Quantum Foundations · Module 8 — Open Systems

Node 8.5

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Open Quantum Systems **Node:** 8.5

A **decoherence-free subspace (DFS)** is a sector of a system's Hilbert space on which a particular noise channel acts trivially — every state in the subspace is preserved under the noise. Quantum information encoded into such a subspace is *passively* protected: no active error detection or correction is required, because the noise simply has no effect within the protected sector. DFS are the simplest possible form of noise immunity, and they exploit symmetries of the noise model to identify "blind spots" of the environment.

DFS are conceptually older and structurally simpler than the active quantum error correction codes of Node 8.6. They are also more fragile: a DFS protects against the *specific* noise model it is constructed for, and provides no protection against other forms of noise. In practice, DFS encoding is used when the dominant noise channel has a clean symmetry (for example, collective dephasing of qubits in a region of uniform magnetic field), while active error correction is used for generic noise. The two strategies are complementary: combining DFS encoding with active error correction can yield substantial improvements in effective fidelity.

Formal Definition

Decoherence-free subspace

Let $\{L_k\}$ be the Lindblad operators of an open-system master equation, and H the system Hamiltonian. A subspace $\mathcal{H}_{\mathrm{DFS}} \subseteq \mathcal{H}$ is **decoherence-free** with respect to this dynamics if:

1. For every $|\psi\rangle \in \mathcal{H}_{\mathrm{DFS}}$ and every k, each Lindblad operator acts as a multiple of the identity:

$$L_k|\psi\rangle = c_k|\psi\rangle,$$

   for some constants c_k independent of $|\psi\rangle$.

2. The Hamiltonian preserves $\mathcal{H}_{\mathrm{DFS}}$, i.e., $H\mathcal{H}_{\mathrm{DFS}} \subseteq \mathcal{H}_{\mathrm{DFS}}$.

When these conditions hold, the projector P_{DFS} onto the subspace commutes with the dynamics, and the reduced state of any DFS-encoded pure state evolves unitarily inside $\mathcal{H}_{\mathrm{DFS}}$ under an effective Hamiltonian $H_{\mathrm{eff}} = P_{\mathrm{DFS}} H P_{\mathrm{DFS}} - \frac{i\hbar}{2}\sum_k |c_k|^2 \mathbb{1}_{\mathrm{DFS}}$, with the second term contributing only an overall phase. The subspace is invariant under the open-system dynamics and the encoded states do not decohere.

Equivalent characterization via Lindbladian kernel

A DFS can equivalently be characterized as a subspace on which the dissipator vanishes. Compute

$$\mathcal{D}[L_k](|\psi\rangle\langle\psi|) = L_k|\psi\rangle\langle\psi|L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, |\psi\rangle\langle\psi|\}.$$

If $L_k|\psi\rangle = c_k|\psi\rangle$, this becomes

$$|c_k|^2|\psi\rangle\langle\psi| - |c_k|^2|\psi\rangle\langle\psi| = 0.$$

So the dissipator vanishes on DFS states, confirming that they evolve only under the unitary part of the dynamics.

Necessary and sufficient conditions

Theorem (Lidar–Chuang–Whaley, 1998). A subspace $\mathcal{H}_{\text{DFS}}$ is decoherence-free with respect to a Lindblad equation iff:

1. Each L_k restricted to $\mathcal{H}_{\text{DFS}}$ is proportional to the identity on the subspace.
2. H restricted to $\mathcal{H}_{\text{DFS}}$ maps the subspace into itself (no leakage).
3. (Sometimes added for cleanliness) The subspace is contained in a common eigenspace of $\{L_k^\dagger L_k\}$.

For *strict* DFS the constants c_k must be the same for all states in the subspace; if they vary across the subspace, the dynamics is unitary on the subspace but is no longer strictly noise-free in the usual sense (one then enters the territory of *noiseless subsystems*, a strict generalization of DFS).

Noiseless subsystems

A more general notion is the **noiseless subsystem (NS)**: the system's Hilbert space decomposes as $\mathcal{H} = \bigoplus_J \mathcal{H}_J^L \otimes \mathcal{H}_J^N$, and noise acts only on the L factor (the "logical" part is left unchanged). A DFS is the special case where the N factor is one-dimensional. Noiseless subsystems can encode quantum information immune to a wider class of noises than DFS, at the cost of more complex encoding.

Intuition

The intuition behind DFS is symmetry. If the noise has a symmetry that the encoded states all share, the noise cannot distinguish among them, and therefore cannot decohere them. The simplest example is **collective dephasing** of two qubits: a noise process that flips the phase of both qubits identically. The two states $|01\rangle$ and $|10\rangle$ both have one excitation, so they are related by a permutation symmetry that the collective phase noise respects. Phase shifts on both qubits act as global phases on these states, leaving the relative phase unchanged. Therefore the subspace spanned by $|01\rangle$ and $|10\rangle$ is decoherence-free against collective dephasing.

The general principle: identify a symmetry that the noise model respects, find a subspace whose states transform trivially (or all the same way) under that symmetry, and encode logical qubits in that subspace. The encoding is "passive" because no measurement, no syndrome extraction, no correction is needed — the symmetry of the noise itself guarantees that the encoded states are preserved.

The cost is fragility. A DFS protects only against the specific symmetry-respecting noise it was designed for. Any noise that breaks the symmetry — say, individual qubit dephasing on top of collective dephasing — will still affect the encoded states. In real hardware, noise is typically a mixture of collective and individual processes, and a pure DFS encoding only protects against the collective component. For full protection one needs active error correction (Node 8.6).

The structural role of DFS is also as a stepping stone: every active error-correcting code can be viewed as a DFS for the *encoded* noise model after a recovery operation. The Knill–Laflamme conditions for quantum error correction are a generalization of the DFS conditions to allow active recovery. Understanding DFS is therefore a prerequisite for understanding why and how quantum error correction works.

Structural Role

- **Passive noise protection.** DFS provide noise protection without active intervention. This is attractive when the noise model has a clean symmetry and when active error correction is expensive (requires extra qubits, gates, and measurements).
- **Foundation for active error correction.** The conditions for a subspace to be a DFS are a special case of the Knill–Laflamme conditions for an error-correcting code: in DFS the recovery operation is the identity, while in active codes the recovery is a nontrivial unitary or measurement-conditioned operation.
- **Practical applications.** Trapped-ion quantum computers use DFS encoding to protect against collective dephasing from magnetic field fluctuations, which is the dominant noise in many ion-trap systems. NV-center qubits use similar tricks to protect against collective fluctuations in nuclear spin baths.
- **Symmetry-protected logical qubits.** The DFS framework gives a way to think about logical qubits as transformations under specific symmetry groups. The symmetric and antisymmetric subspaces of n qubits under permutations are the prototypical examples.
- **Connection to subsystem codes.** The Bacon–Shor code and other subsystem codes use the noiseless-subsystem generalization of DFS to protect against more complex noise models with simpler encoding circuits.

Mathematical Development

Collective dephasing on n qubits

Consider n qubits all coupled to the same dephasing noise via the collective Lindblad operator

$$L = \sum_{i=1}^{n} Z_i.$$

The eigenspaces of L are the "Dicke" sectors with definite S_z total. Within each Dicke sector, all states have $L|\psi\rangle = m|\psi\rangle$ with the same eigenvalue m, so the Lindblad operator acts as a multiple of identity. Therefore each Dicke sector is a DFS for collective dephasing.

For $n = 2$, the Dicke sectors are:

- $m = +2$: $\{|00\rangle\}$ (1-dimensional, trivial DFS).
- $m = 0$: $\{|01\rangle, |10\rangle\}$ (2-dimensional DFS — encodes 1 logical qubit).
- $m = -2$: $\{|11\rangle\}$ (1-dimensional).

The 2-dimensional DFS at $m = 0$ encodes a single logical qubit. The logical $|0_L\rangle = |01\rangle$ and $|1_L\rangle = |10\rangle$ are immune to collective dephasing — any superposition $\alpha|01\rangle + \beta|10\rangle$ acquires only an overall phase under the noise (which has no observable effect).

For $n = 4$, the $m = 0$ sector has dimension $\binom{4}{2} = 6$, so it can encode $\log_2 6 \approx 2.58$ logical qubits. In general, the dimension of the $m = 0$ sector for $2k$ qubits is $\binom{2k}{k}$, giving an encoding rate that approaches 1 logical qubit per physical qubit asymptotically.

Collective spin-flip

For collective X-noise, $L = \sum_i X_i$, the DFS is constructed similarly using eigenstates of $S_x = \frac{1}{2}\sum_i X_i$. The 2-qubit DFS for collective spin-flip is the antisymmetric singlet $|s\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ and its complement, with appropriate logical qubit encoding.

General collective noise

For a noise model with collective Lindblad operator $L = \sum_i A_i$, where each A_i acts on qubit i, the DFS structure is determined by the symmetric group S_n: subspaces transforming trivially under the collective action are DFS, and the dimensions of DFS sectors are computed via representation theory of S_n.

For arbitrary collective noise (combinations of X, Y, Z acting collectively on all qubits), the DFS is the **singlet sector** of total spin $J = 0$, which exists only for an even number of qubits. The minimal singlet-DFS encoding uses 4 qubits to encode 1 logical qubit (DFS dimension 2) protected against arbitrary collective noise. This is the Zanardi–Rasetti construction.

Knill–Laflamme conditions for DFS

The Knill–Laflamme conditions for an error-correcting code with code projector P and error operators $\{E_a\}$ state that:

$$PE_a^\dagger E_b P = \alpha_{ab} P$$

for some matrix α_{ab}. When $\alpha_{ab} = \delta_{ab} \cdot \text{const}$, the errors take the code into orthogonal subspaces and can be corrected. When α_{ab} is a multiple of identity *and* $E_a P = c_a P$ for each a, the errors do not take the code out of itself — the code is a DFS.

This shows that DFS is the "trivial recovery" case of quantum error correction: no syndrome measurement, no active recovery, just the natural protection of the encoded subspace.

Worked Example

2-qubit DFS for collective dephasing

Consider two qubits with collective dephasing Lindblad operator $L = Z_1 + Z_2$. The eigenstates of L and their eigenvalues:

- $|00\rangle$: eigenvalue $+2$
- $|01\rangle$: eigenvalue 0
- $|10\rangle$: eigenvalue 0
- $|11\rangle$: eigenvalue -2

The 2-dimensional eigenspace at eigenvalue 0 — spanned by $|01\rangle, |10\rangle$ — is the DFS. Encode a logical qubit:

$$|0_L\rangle = |01\rangle, \qquad |1_L\rangle = |10\rangle.$$

A general logical state is $|\psi_L\rangle = \alpha|01\rangle + \beta|10\rangle$.

Check noise immunity. Apply the dissipator: $L|\psi_L\rangle = 0 \cdot |\psi_L\rangle = 0$, so

$$\mathcal{D}[L](|\psi_L\rangle\langle\psi_L|) = 0 \cdot 0 - \frac{1}{2}\{0, |\psi_L\rangle\langle\psi_L|\} = 0.$$

The state evolves only under the Hamiltonian. If the Hamiltonian is $H = \omega(Z_1 + Z_2)/2$ (collective drive), then $H|\psi_L\rangle = 0$ (since L is proportional to H here), and the encoded state is completely frozen. To enable logical operations, one needs Hamiltonian terms that act non-trivially within the DFS, such as $X_1X_2 + Y_1Y_2$, which exchanges $|01\rangle \leftrightarrow |10\rangle$ and acts as a logical X gate.

Singlet-triplet qubit in semiconductor double quantum dots

In semiconductor double-dot systems, two electrons can be encoded in singlet $|s\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ and triplet $|t_0\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ states. The two-state subspace $\{|s\rangle, |t_0\rangle\}$ is decoherence-free against collective noise (uniform magnetic field fluctuations on both dots) and forms the "S-T0 qubit" used in many semiconductor implementations. Logical operations are performed via exchange interactions and gradient magnetic fields.

4-qubit DFS for arbitrary collective noise

For protection against arbitrary collective noise ($L_X = X_1 + X_2 + X_3 + X_4,\ L_Y = Y_1 + Y_2 + Y_3 + Y_4,$ $L_Z = Z_1 + Z_2 + Z_3 + Z_4$), the DFS is the total-spin-zero (singlet) subspace of 4 qubits. This subspace has dimension 2 and encodes one logical qubit. The logical states are:

$$|0_L\rangle = \frac{1}{2}(|0011\rangle - |0110\rangle - |1001\rangle + |1100\rangle), \qquad |1_L\rangle = \frac{1}{\sqrt{12}}(2|0101\rangle - |0110\rangle + 2|1010\rangle - |1001\rangle - |0011\rangle - |1100\rangle).$$

(Up to choice of basis within the singlet sector.) These logical states are completely immune to any noise that acts as a collective operator on all four qubits. They are not immune to individual qubit errors, which are not collective. This is the Zanardi–Rasetti DFS.

Failure under independent noise

If the noise on each qubit is independent rather than collective, the DFS fails. For example, with two qubits and independent dephasing $L_1 = Z_1, L_2 = Z_2$, the state $|01\rangle$ is no longer an eigenstate of both Lindblad operators (it is an eigenstate of Z_1 with eigenvalue $+1$ but of Z_2 with eigenvalue -1, not the same constant). The dissipator does not vanish, and the encoded state decoheres. This illustrates the fragility of DFS to symmetry-breaking noise.

Cross-Links

- **Module 8.2 (Lindblad equation):** DFS conditions are formulated in terms of the Lindblad operators. The dissipator vanishes on DFS states.
- **Module 8.6 (Quantum error correction):** Active error correction generalizes DFS by allowing nontrivial recovery operations. The Knill–Laflamme conditions reduce to DFS conditions when the recovery is the identity.
- **Module 5.5 (Pointer basis / einselection):** Both DFS and pointer states are subspaces preserved by the system-environment coupling, but they differ in what is preserved: pointer states are classical-like states selected by decoherence, while DFS are arbitrary quantum states protected by symmetry.
- **Module 4.6 (Monogamy of entanglement):** DFS encoding distributes the logical state across multiple physical qubits, exploiting the structure of multi-particle entanglement.
- **Statistics — symmetry-protected statistics:** Classical analogue is the use of conserved quantities (e.g., total spin, total particle number) to reduce the effective state space and protect against generic noise that respects the conservation law.

Structural Compression

Invariant: A subspace $\mathcal{H}_{\mathrm{DFS}}$ is decoherence-free iff every Lindblad operator acts on it as a multiple of identity, and the Hamiltonian preserves the subspace. Within $\mathcal{H}_{\mathrm{DFS}}$, the dissipator vanishes and states evolve unitarily under an effective Hamiltonian.

Mechanism: Identify a symmetry of the noise model; find a subspace on which all noise operators act trivially; encode logical information in that subspace. The encoding is passive — no syndrome extraction or active recovery is needed.

Cross-System Echo: Symmetry-protected ground states in condensed matter; conserved-quantity sectors in classical mechanics; anyonic encoding in topological quantum computing — all use the same structural pattern of "encode in a sector that the noise cannot reach."

Exercises

1. Verify that the 2-dimensional subspace spanned by $|01\rangle, |10\rangle$ is a DFS for collective dephasing $L = Z_1 + Z_2$. Compute the dissipator on a general state in this subspace.
2. Construct logical operations on the 2-qubit collective-dephasing DFS using Hamiltonian terms that preserve the subspace.
3. Show that the singlet-triplet qubit $\{|s\rangle, |t_0\rangle\}$ is a DFS for collective noise.
4. Determine the dimension of the maximal DFS for collective dephasing on n qubits, and identify the encoding rate as $n \to \infty$.
5. Explain why DFS encoding fails when noise is independent rather than collective. Quantify the effective dephasing rate of an encoded logical qubit under weak independent noise.
6. State the Knill–Laflamme conditions for a quantum error-correcting code, and show how DFS is the special case where the recovery is the identity operation.
7. Argue, structurally, why DFS is a "free lunch" in the appropriate noise regime, and discuss the cost (encoding overhead, sensitivity to noise model assumptions, restricted Hamiltonian set) that makes it not always the right choice.

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Concepts: decoherence, hilbert-space, quantum-noise
Prerequisites: 8.2
Enables: 8.6

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