Landauer's Principle

Quantum Foundations · Module 9 — Quantum Thermodynamics

Node 9.4

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Quantum Thermodynamics **Node:** 9.4

Landauer's principle is the structural bridge between information theory and thermodynamics: any logically irreversible operation — paradigmatically, erasing a bit — has a non-zero minimum thermodynamic cost, $k_B T \log 2$ of heat per bit dissipated into the environment. It is the reason Maxwell's demon does not violate the second law, and the reason classical computation has a fundamental energetic floor. In the quantum setting it follows directly from the strong subadditivity of von Neumann entropy (7.4) together with the Gibbs-state fixed-point property (9.1).

Formal Statement

Landauer's principle (Landauer 1961; quantum formulation: Reeb–Wolf 2014): Let a memory M (system of interest) be reset by a protocol that couples it to a thermal bath B at inverse temperature $\beta = 1/(k_B T)$. Let the joint evolution be unitary on MB, with B initially in the Gibbs state $\rho_B^{\mathrm{th}} = e^{-\beta H_B}/Z_B$ and uncorrelated with M. Then the average heat dissipated into the bath,

$$\langle Q \rangle = \mathrm{tr}\big((\rho'_B - \rho_B^{\mathrm{th}})H_B\big)\,,$$

is bounded below by the entropy decrease of the memory:

$$\boxed{\beta\langle Q\rangle \geq S(\rho_M) - S(\rho'_M)}$$

where ρ_M, ρ'_M are the initial and final reduced memory states and S is von Neumann entropy. In particular, if the protocol resets the memory from a maximally mixed 1-bit state $\rho_M = I/2$ to a pure state $\rho'_M = |0\rangle\langle 0|$,

$$\langle Q \rangle \geq k_B T \log 2.$$

The Reeb–Wolf refinement replaces $\geq$ with a finite-size equality plus non-negative correction terms from residual MB correlations and bath non-thermality after the protocol. Equality with the Landauer bound is reached only in the thermodynamic limit with quasi-static erasure.

Intuition

A "bit" is a memory whose state-space has two distinguishable configurations. Erasing it means *resetting* all configurations to a single one, say $|0\rangle$. This is an entropy-decreasing operation on the memory — from 1 bit of entropy down to 0. The second law says total entropy can only increase, so something else must absorb at least 1 bit. The only available sink is the thermal bath, and the thermodynamic “currency” for 1 bit of entropy at temperature T is $k_B T \log 2$ joules.

The *why* is structural:

1. The memory’s entropy can only decrease if the bath’s entropy increases.
2. The bath’s entropy can only increase if it absorbs heat.
3. The minimum heat required for a given entropy increase is set by $dS = dQ/T$ at the bath temperature.

Combining these three gives Landauer. It is not a statement about any particular physical implementation — it is a statement about what any physically realizable erasure *must* do to the total entropy. Maxwell’s demon fails because the demon’s memory must eventually be erased, and that erasure cost is exactly what the demon “saved” earlier.

Structural Role

Landauer’s principle anchors the identification of Shannon entropy (bits) with thermodynamic entropy (k_B per nat). Without it the two would be formally analogous but dimensionally disconnected. With it they become physically the same quantity, up to a unit conversion $k_B \log 2$ per bit.

Downstream:

- **9.5 Quantum heat engines** inherit Landauer as a lower bound on any cycle that resets a memory register.
- **9.6 Resource theories** of thermodynamics count “pure states” as thermodynamic resources whose exchange rate with heat is $k_B T \log 2$ per bit.
- **Reversible computation** (Bennett 1973) is the logical converse: a computation without bit erasure has no Landauer cost, only finite-time dissipation.
- **Black-hole thermodynamics** uses a Landauer-like identification when assigning entropy $k_B \log 2$ per bit to horizon area.

Mathematical Development

Proof via second law on memory + bath

Let the joint state evolve unitarily: $\rho'_{MB} = U \rho_M \otimes \rho_B^{\text{th}} U^\dagger$. Unitary evolution preserves total entropy:

$$S(\rho'_{MB}) = S(\rho_{MB}) = S(\rho_M) + S(\rho_B^{\text{th}}).$$

By subadditivity of von Neumann entropy (7.4),

$$S(\rho'_{MB}) \le S(\rho'_M) + S(\rho'_B).$$

Combine:

$$S(\rho_M) + S(\rho_B^{\text{th}}) \le S(\rho'_M) + S(\rho'_B).$$

Rearrange:

$$S(\rho_M) - S(\rho'_M) \le S(\rho'_B) - S(\rho_B^{\text{th}}).$$

Now use the fact that the Gibbs state maximizes entropy at fixed expected energy. Concretely, Klein’s inequality applied to ρ'_B and ρ_B^{th} gives

$$S(\rho_B') - S(\rho_B^{\text{th}}) \le \beta \operatorname{tr}\big((\rho_B' - \rho_B^{\text{th}})H_B\big) = \beta\langle Q\rangle.$$

Chaining the two inequalities:

$$S(\rho_M) - S(\rho_M') \le \beta\langle Q\rangle. \qquad \blacksquare$$

Where each step is used

- **Unitarity of joint evolution**: ensures total entropy is conserved, so memory-entropy decrease = bath-entropy increase.
- **Subadditivity**: accounts for any residual correlations between M and B after the protocol. Equality requires the post-protocol state to be product (uncorrelated memory + bath).
- **Gibbs being the max-entropy state**: ensures heat-to-entropy conversion at the bath achieves at best the rate $1/k_BT$. Equality requires the bath to remain thermal after erasure.

Both equality conditions can only be met in the idealized quasi-static, thermodynamic-limit case. Real erasures have a finite-size Reeb–Wolf correction.

Reeb–Wolf equality

Reeb and Wolf (2014) proved the exact version:

$$\beta\langle Q\rangle = S(\rho_M) - S(\rho_M') + I(M' : B') + D(\rho_B' \| \rho_B^{\text{th}}),$$

where $I(M' : B')$ is the mutual information between memory and bath after the protocol and D is the relative entropy. Both correction terms are non-negative, so Landauer's bound is recovered, and it shows explicitly what needs to vanish for equality.

Worked Example

Erasing a single qubit memory

Let M be a single qubit with trivial internal Hamiltonian (degenerate), initially in $\rho_M = I/2$ (maximally mixed, 1 bit of entropy). The bath is a large thermal reservoir at temperature T. The erasure protocol:

1. **Couple** M to an auxiliary qubit with Hamiltonian $H_M = \epsilon|1\rangle\langle 1|$, adiabatically raised from $\epsilon = 0$ to $\epsilon \gg k_BT$. Under slow coupling + bath contact, M equilibrates at each ϵ to $e^{-\beta H_M}/Z$.
2. At $\epsilon \gg k_BT$, the Gibbs state is $\approx |0\rangle\langle 0|$ — the memory is erased.
3. **Decouple** by lowering $\epsilon \to 0$ again. If done slowly, the state remains $|0\rangle\langle 0|$.

Work done on M during the ramp-up (reversible, integrated over ϵ):

$$W_{\text{ramp}} = \int_0^\infty \partial_\epsilon F(\epsilon)\, d\epsilon = \int_0^\infty \frac{\epsilon\, e^{-\beta\epsilon}}{1 + e^{-\beta\epsilon}}\, d\epsilon \cdot \beta\text{-factor},$$

which evaluates to exactly $k_BT \log 2$ in the quasi-static limit.

Heat dissipated to the bath: $\langle Q\rangle = k_BT\log 2$ (by the first law, since $\Delta U_M = 0$ for the cycle $0 \to \epsilon \to 0$).

Numerical value at $T = 300$ K:

$$k_BT\log 2 \approx (1.38 \times 10^{-23}\,\text{J/K})(300\,\text{K})(0.693) \approx 2.87 \times 10^{-21}\,\text{J} \approx 17.9\,\text{meV}.$$

Modern CMOS gates dissipate $\sim 10^{-18}$ J per bit operation, still three orders of magnitude above Landauer's bound. Recent experiments (Bérut et al. 2012 classical; Yan et al. 2018 quantum Josephson) have directly measured heat flow during bit erasure and verified the bound to within the Reeb–Wolf correction terms.

Cross-Links

- **7.4 Von Neumann entropy** — supplies subadditivity $S(\rho_{MB}) \leq S(\rho_M) + S(\rho_B)$ and Klein's inequality.
- **9.1 Gibbs state** — the max-entropy characterization gives the $\leq \beta\langle Q\rangle$ step.
- **9.2 Work and heat** — identifies what $\langle Q\rangle$ *is* in the TPM scheme.
- **9.5 Quantum heat engines** — any cycle that includes a reset step pays a Landauer cost.
- **Information theory** — identifies Shannon bit with physical bit; Maxwell's demon resolution (Bennett 1982).
- **Reversible computation** — Bennett/Fredkin/Toffoli gates avoid erasure and thus Landauer cost.

Structural Compression

Invariant: Erasing ΔS nats of memory entropy requires dissipating at least $\Delta S/\beta$ joules of heat into the bath.

Mechanism: Subadditivity of joint entropy + Klein's inequality against the Gibbs state.

Cross-System Echo: Same form in classical Landauer (derived from Shannon entropy) and quantum Landauer (derived from von Neumann entropy). The structural identity $S_{\text{info}} \leftrightarrow S_{\text{thermo}}$ is the content of the theorem; everything else is bookkeeping.

Exercises

1. Prove that if the joint post-protocol state ρ'_{MB} is uncorrelated ($I(M' : B') = 0$) and the bath remains thermal ($D(\rho'_B \| \rho_B^{\text{th}}) = 0$), then Landauer's inequality is saturated. Interpret each condition physically.

2. Compute the Landauer cost of erasing an n-bit memory that starts in a state of Shannon entropy H bits and ends in the all-zero state. Show the answer is $n \log 2 - H$ in units of $k_B T$, *not* $n \log 2$.

3. Derive Klein's inequality $S(\rho) - S(\sigma) \leq \operatorname{tr}(\rho - \sigma) \log \sigma^{-1}$ from the non-negativity of quantum relative entropy, and use it to justify the $\beta\langle Q\rangle$ step in the proof.

4. A Maxwell demon observes a single-molecule gas and records which half of the container the molecule is in, then uses this to extract $k_B T \log 2$ of work. Show that the demon's memory must hold 1 bit, and that erasing this memory at the end of the cycle costs exactly $k_B T \log 2$, restoring the second law.

5. A quantum erasure protocol takes τ seconds and dissipates $k_B T \log 2 + \Delta$ per bit, with $\Delta > 0$ due to finite-time effects. Show that, to leading order in $1/\tau$, $\Delta \propto 1/\tau$ and identify the prefactor in terms of the relaxation time of the bath.

6. For a memory with non-degenerate Hamiltonian H_M, show that the Landauer bound generalizes to $\beta\langle Q\rangle \geq S(\rho_M) - S(\rho'_M) - \beta \Delta U_M$, where $\Delta U_M = \operatorname{tr}(\rho'_M H_M) - \operatorname{tr}(\rho_M H_M)$. Interpret the new term as work done on the memory.

7. **Argue, structurally**, why Landauer's principle is the *unique* way to reconcile reversible microdynamics (unitary quantum mechanics) with irreversible logical operations (bit erasure) while preserving the second law. What would break if the coefficient $k_B T \log 2$ were replaced by anything smaller?

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Concepts: von-neumann-entropy, shannon-entropy, information-theory
Prerequisites: 9.1, 7.4

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