Quantum Heat Engines

Quantum Foundations · Module 9 — Quantum Thermodynamics

Node 9.5

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Quantum Thermodynamics **Node:** 9.5

A quantum heat engine extracts work from temperature differences using a quantum system as its working substance. The cycle structure is the same as a classical engine — isothermal contact with a hot bath, expansion, isothermal contact with a cold bath, compression — but the expansions and compressions are protocol-driven changes of a *quantum* Hamiltonian and the equilibrations use a *Lindblad* thermalization from 8.2. The question of node 9.5 is: do quantum effects change the efficiency? The structural answer is **no at the asymptotic Carnot limit, yes at finite-time power and in specific non-equilibrium regimes**.

Formal Definition

A **quantum heat engine** is a cyclic protocol on a quantum system S with a parameter-dependent Hamiltonian $H(\lambda)$, alternately coupled to a hot bath at temperature T_h and a cold bath at temperature T_c. Over one cycle, the state returns to itself: $\rho_{\text{end}} = \rho_{\text{start}}$. By the first law applied to a cycle,

$$\oint d\langle E\rangle = 0 \quad \Longrightarrow \quad W_{\text{net}} + Q_h + Q_c = 0,$$

where $Q_h > 0$ is heat absorbed from the hot bath, $Q_c < 0$ is heat dumped to the cold bath, and $W_{\text{net}} < 0$ is work extracted (with the sign convention that work done *on* the system is positive).

Efficiency is defined as

$$\eta = \frac{|W_{\text{net}}|}{Q_h} = 1 - \frac{|Q_c|}{Q_h}.$$

Carnot bound: For any cyclic engine operating between $T_h > T_c$,

$$\eta \leq \eta_C = 1 - \frac{T_c}{T_h}.$$

Equality holds only for reversible (quasi-static) cycles. This bound follows from the Clausius form of the second law $\oint dQ/T \leq 0$, which survives unchanged in the quantum setting because it follows from the Gibbs fixed-point property of Lindblad thermalization (9.1 + 8.2).

Intuition

Think of a working substance with a tunable energy gap ω. When you couple it to the hot bath, it fills up with thermal populations characteristic of T_h, ω_h. When you decouple it and compress (raise ω), you do work on it without changing populations — its internal energy goes up. When you couple to the cold bath, it sheds heat until equilibrated at T_c, ω_h. Finally, decouple and expand (lower ω) to close the cycle, extracting work.

The net effect over a full cycle: heat flows from hot to cold, and a fraction $\eta \leq \eta_C$ of the hot-side heat comes out as work. The "quantumness" enters in:

- The working substance is quantized — it's a qubit or oscillator, not a classical gas.
- Between strokes, the system can be in superpositions or have coherences.
- Coherences can be exploited (squeezed-bath engines, coherence-enhanced engines) but only within the Carnot constraint imposed by the second law.

There is no such thing as a quantum engine that beats Carnot. If a paper claims otherwise, it is tacitly redefining one of T_h, T_c, W, or Q.

Structural Role

9.5 is the operational arena for testing quantum thermodynamic concepts in the lab. Every theoretical tool from this module is used:

- The **Gibbs state** (9.1) sets the equilibrium endpoints.
- The **TPM work distribution** (9.2) gives statistics per stroke.
- **Fluctuation theorems** (9.3) constrain finite-time dissipation.
- **Landauer's principle** (9.4) applies if the cycle involves bit reset.
- **Lindblad thermalization** (8.2) models the bath contact strokes.

Quantum heat engines have been built and run: from trapped-ion heat engines (Roßnagel et al. 2016) to NMR engines (Peterson et al. 2019) to superconducting-circuit heat engines. Efficiencies up to ~30% of Carnot have been measured; the deviation is entirely due to finite-time dissipation, not any violation of the Carnot bound.

Mathematical Development

Quantum Otto cycle on a qubit

The Otto cycle has four strokes. Let the qubit have $H(\lambda) = \frac{\lambda}{2} Z$, with λ tunable between ω_c and ω_h, and $\omega_h > \omega_c$.

Stroke 1: Isochoric hot ($\lambda = \omega_h$, coupled to T_h). Start in ρ_A, end in $\rho_B = e^{-\beta_h \omega_h Z/2}/Z_{h,B}$. Heat absorbed:

$$Q_1 = \mathrm{tr}(\rho_B H_h) - \mathrm{tr}(\rho_A H_h).$$

Stroke 2: Adiabatic compression/expansion ($\lambda : \omega_h \to \omega_c$, decoupled from bath). The state follows U_{ad}. For the qubit, adiabaticity means populations in $H(\lambda)$ eigenstates are preserved. No heat, only work:

$$W_2 = \mathrm{tr}(\rho_B H_c) - \mathrm{tr}(\rho_B H_h).$$

Since populations don't change and the energy gap shrinks ($\omega_h \to \omega_c$), $W_2 < 0$ (work out).

Stroke 3: Isochoric cold ($\lambda = \omega_c$, coupled to T_c). End in $\rho_D = e^{-\beta_c \omega_c Z/2}/Z_{c,D}$. Heat released to cold bath:

$$Q_3 = \mathrm{tr}(\rho_D H_c) - \mathrm{tr}(\rho_B H_c) < 0.$$

Stroke 4: Adiabatic compression ($\lambda : \omega_c \to \omega_h$).

$$W_4 = \mathrm{tr}(\rho_D H_h) - \mathrm{tr}(\rho_D H_c) > 0.$$

By cycle closure, $\rho_D \to \rho_A$ under the compression, so the equilibration at T_h restores ρ_B with the same populations as ρ_A carried over from ρ_D plus bath relaxation.

Otto efficiency

The quantum Otto efficiency for a qubit between (T_h, ω_h) and (T_c, ω_c) is

$$\eta_{\mathrm{Otto}} = 1 - \frac{\omega_c}{\omega_h}.$$

Compare with Carnot $\eta_C = 1 - T_c/T_h$. The Otto cycle reaches Carnot only when

$$\frac{\omega_c}{\omega_h} = \frac{T_c}{T_h},$$

i.e. when the compression ratio matches the temperature ratio. Otherwise $\eta_{\mathrm{Otto}} < \eta_C$, and the engine is strictly sub-Carnot even though its specific efficiency formula looks “simpler.”

At the Carnot limit the cycle reduces to zero net power (infinite cycle time). The curve-Novikov-Chambadal bound at **maximum power** is $\eta_{\mathrm{MP}} = 1 - \sqrt{T_c/T_h}$, and this is the more relevant figure of merit for real devices.

Why the Carnot bound survives quantization

The Carnot bound comes from applying the Clausius inequality $\oint dQ/T \leq 0$ to the cycle. In the quantum case this becomes

$$\sum_{\mathrm{baths}} \int \frac{d\langle Q_{\mathrm{bath}} \rangle}{T_{\mathrm{bath}}} \leq 0,$$

which follows from strong subadditivity of von Neumann entropy + the Gibbs state being the max-entropy state. There is no step in the derivation that uses commutativity, classicality, or absence of superposition. Hence the bound survives.

Worked Example

Trapped-ion Otto engine (Roßnagel et al. 2016 style)

A single $^{40}Ca^+$ ion in a tapered Paul trap acts as the working substance. The “gap” is the spacing of vibrational levels, tuned by the trap frequency $\nu(t)$ between $\nu_h = 1$ MHz and $\nu_c = 200$ kHz. Hot bath: electric-field noise injected at $T_h \sim 10^4$ K effective temperature. Cold bath: spontaneous emission at $T_c \sim 10^2$ K.

Temperature ratio: $T_c/T_h = 0.01$, so $\eta_C \approx 0.99$.

Frequency ratio: $\nu_c/\nu_h = 0.2$, so $\eta_{\mathrm{Otto}} = 0.8$.

Measured efficiency: ~0.28, limited by finite-time dissipation, phonon leakage, and imperfect thermalization. The measurement verified the structural prediction: the cycle performs as a heat engine, efficiency obeys the Otto formula as an upper bound, and Carnot is never approached because the cycle is fast relative to the relaxation time.

Interpretation in the TPM framework: work per stroke is a random variable; only the average obeys the clean formulas above. Individual runs exhibit the full Crooks distribution.

Zero-power Carnot limit

Take $\omega_h \to \omega_c$ at fixed T_h, T_c. The gap difference vanishes, so the adiabatic strokes do zero work per cycle. But the isothermal strokes also transfer zero net heat because the populations at T_h, ω and T_c, ω differ only by a factor $\sim (\beta_h - \beta_c)\omega$. The ratio $\eta \to 1 - T_c/T_h$ is recovered by quasi-static integration. This illustrates the general rule: high efficiency $\leftrightarrow$ low power.

Cross-Links

- **9.1 Gibbs state** — thermalization endpoints of each isothermal stroke.
- **9.2 TPM work** — stochastic work per stroke.
- **9.3 Fluctuation theorems** — constrain dissipation in finite-time cycles.
- **9.4 Landauer** — applies if the engine includes any memory reset.
- **8.2 Lindblad thermalization** — models the bath-contact strokes.
- **5.1 Density matrices** — the engine state is generally mixed during the cycle.
- **Classical thermodynamics** — Otto, Carnot, Stirling cycles all have direct quantum analogues.

Structural Compression

Invariant: $\eta \le \eta_C = 1 - T_c/T_h$ for any cyclic engine between two thermal baths, quantum or classical.

Mechanism: Clausius inequality applied via strong subadditivity + Gibbs max-entropy.

Cross-System Echo: Classical Carnot (1824). Quantum Otto, quantum Carnot, quantum Stirling — all obey the same bound. Quantum coherence can shift *power*, not *efficiency limit*.

Exercises

1. Derive the Otto efficiency $\eta_{\text{Otto}} = 1 - \omega_c/\omega_h$ for a qubit working substance, starting from the per-stroke heat calculations.

2. Show that a quantum Carnot cycle (two isothermals + two adiabatics, all quasi-static) reaches η_C in the quasi-static limit, and that the net power vanishes as τ^{-1}.

3. For a quantum harmonic-oscillator Otto cycle with $H(\lambda) = \hbar\lambda(a^\dagger a + 1/2)$, compute Q_h, Q_c, W_{net} in terms of $\lambda_h, \lambda_c, T_h, T_c$, and verify $\eta = 1 - \lambda_c/\lambda_h$.

4. Show that a coherence-enhanced engine (starting the stroke in a superposition of energy eigenstates) has the same average work output as a diagonal one *to first order*, but non-zero work *variance*. Relate this to the linear-response limit of Crooks.

5. Compute the Curzon–Ahlborn efficiency at maximum power $\eta_{\text{MP}} = 1 - \sqrt{T_c/T_h}$ for a quantum Otto engine in the high-temperature limit, and show that it is always less than η_C but can exceed $\eta_C/2$.

6. A squeezed-bath engine uses a bath whose state is $e^{-\beta H_{\text{bath}}}/Z$ squeezed by a Bogoliubov transformation. Show that it can *appear* to exceed Carnot if one misidentifies the effective temperature, and correct the analysis by computing the actual entropy production.

7. **Argue, structurally**, why the Carnot bound is the same in classical and quantum thermodynamics. Identify the step in the derivation where classical commutativity *could* in principle matter, and explain why it does not.

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Concepts: density-matrix, quantum-channels
Prerequisites: 9.1, 9.2

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