Resource Theories

Quantum Foundations · Module 9 — Quantum Thermodynamics

Node 9.6

Position in Knowledge Graph

System: Quantum Foundations **Structural Domain:** Quantum Thermodynamics **Node:** 9.6

A **resource theory** is a framework in which certain states and certain operations are declared "free," and everything else is quantified as a costly resource relative to that choice. Quantum thermodynamics becomes a resource theory when the free states are Gibbs states and the free operations are *thermal operations*. The resulting framework generalizes the second law from a single inequality into an infinite family of monotones — one for each Rényi divergence — and gives an exact, non-asymptotic characterization of which non-equilibrium states can be converted into which. This is the modern foundation of quantum thermodynamics (Brandão, Horodecki, Oppenheim, Renes, Spekkens, and others, mid-2010s).

Formal Definition

A **resource theory** is specified by three ingredients:

1. **Free states** $\mathcal{F} \subset \mathcal{S}(\mathcal{H})$: the states you are allowed to have at no cost.
2. **Free operations** $\mathcal{O}$: the CPTP maps you are allowed to apply at no cost. $\mathcal{O}$ must map $\mathcal{F} \to \mathcal{F}$.
3. **Resource monotones** $M : \mathcal{S}(\mathcal{H}) \to \mathbb{R}_{\geq 0}$: functions that are non-increasing under free operations. $M(\rho) \leq M(\Lambda(\rho))\ldots$ sorry, $M(\Lambda(\rho)) \leq M(\rho)$ for all $\Lambda \in \mathcal{O}$.

Thermal resource theory (with respect to Hamiltonian H, temperature T):

- **Free states:** The Gibbs state $\rho_\beta = e^{-\beta H}/Z$ is the *only* free state for each subsystem. (Any other state is a resource called "athermality.")
- **Free operations (thermal operations):** CPTP maps Λ on the system S that admit a Stinespring dilation of the form
$$\Lambda(\rho_S) = \mathrm{tr}_B\left[U(\rho_S \otimes \rho_B^{\mathrm{th}})U^\dagger\right],$$
where $\rho_B^{\mathrm{th}} = e^{-\beta H_B}/Z_B$ is the bath's own Gibbs state and U commutes with the total Hamiltonian $H_S + H_B$ (strict energy conservation).
- **Monotones:** The generalized free energies $F_\alpha(\rho) = \beta^{-1}[S_\alpha(\rho\|\rho_\beta) - \log Z]$ for each Rényi order $\alpha \geq 0$, where $S_\alpha(\rho\|\sigma) = (\alpha-1)^{-1}\log \mathrm{tr}(\rho^\alpha \sigma^{1-\alpha})$. The standard free energy is the $\alpha = 1$ case.

The main theorem: a state ρ can be converted to σ by thermal operations if and only if $F_\alpha(\rho) \geq F_\alpha(\sigma)$ for **all** $\alpha \geq 0$ (quasi-classical case), or equivalently ρ *thermomajorizes* σ.

Intuition

The ordinary second law says "the free energy $F = U - TS$ can only decrease under thermal operations." That gives you *one* inequality. For asymptotically many copies (thermodynamic limit) it is tight: if $F(\rho) > F(\sigma)$, you can convert many copies of ρ into many copies of σ. But for *single shot* or *finite-copy* transformations,

one inequality is too weak. You need an entire family, one per Rényi order, each giving a different constraint on the same transformation.

The reason is structural: the Gibbs state is a complicated object with many "moments" of resourcefulness, and the single-copy free energy only captures the first moment. The higher Rényi divergences capture higher moments (how resource-rich is the state in its tail?), and each generates its own second law. In the asymptotic limit all Rényi divergences converge to the standard relative entropy, and you recover the usual one-line second law.

This is the central structural shift of 21st-century quantum thermodynamics: the second law is not one inequality but a *family*.

Structural Role

9.6 recasts the entire module in a clean, high-level language:

- **9.1 Gibbs state** = the unique free state.
- **9.2 Work** = the athermality of a pure energy-eigenstate resource (the "work bit" $|w\rangle$).
- **9.3 Fluctuation theorems** = consequences of contractivity of relative entropy under thermal operations.
- **9.4 Landauer** = a monotone statement about the $\alpha = 1$ free energy for the erasure transformation.
- **9.5 Heat engines** = cycles in which work is input and output, and the Carnot bound is one inequality from the full family.

It also connects quantum thermodynamics to *other* resource theories:

- **Entanglement** (free = separable states, free = LOCC; monotones = entanglement measures).
- **Coherence** (free = incoherent states, free = incoherent operations; monotones = coherence measures).
- **Magic** (free = stabilizer states, free = Clifford gates; monotones = magic measures).
- **Asymmetry** (free = symmetric states, free = covariant operations).

The resource-theoretic template is the same in all of them. Quantum thermodynamics is one instance among several.

Mathematical Development

Thermomajorization

For a classical (diagonal) state with eigenvalues $p_1, ..., p_d$ at energies $E_1, ..., E_d$, the **thermomajorization curve** is the piecewise-linear function obtained by:

1. Sort the levels by the β-exponential weight $e^{\beta E_i} p_i$ in decreasing order.
2. Plot the cumulative sum of p_i against the cumulative sum of $e^{-\beta E_i}$ on the x-axis.

A state p **thermomajorizes** q, written $p \succ_\beta q$, if its thermomajorization curve lies everywhere on or above q's.

Horodecki–Oppenheim theorem (2013): For quasi-classical states, ρ can be converted to σ by thermal operations if and only if $\rho \succ_\beta \sigma$.

This is the quantum generalization of Hardy–Littlewood–Polya's classical majorization theorem (which recovers the case $\beta = 0$, i.e. $H = 0$).

Generalized free energies

For each Rényi order $\alpha \geq 0$ define

$$F_\alpha(\rho) = \frac{1}{\beta}\left[S_\alpha(\rho\|\rho_\beta) - \log Z\right], \quad S_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log \mathrm{tr}(\rho^\alpha \sigma^{1-\alpha}).$$

Special cases:

- $\alpha = 0$: F_0 — counts the rank of ρ.
- $\alpha = 1$: $F_1 = \mathrm{tr}(\rho H) - TS(\rho)$ — the standard Helmholtz free energy.
- $\alpha = 2$: relates to the second moment / purity.
- $\alpha = \infty$: F_∞ — single-shot worst-case, associated with min-entropy.

Each F_α is monotonically non-increasing under thermal operations. For quasi-classical states, the conjunction of all these inequalities is exactly equivalent to thermomajorization.

Brandão et al. second laws (2015)

The full set of second laws for single-shot thermodynamics:

$$F_\alpha(\rho) \geq F_\alpha(\sigma) \quad \text{for all } \alpha \geq 0.$$

In the asymptotic i.i.d. limit (many copies, averaging), all F_α converge to F_1, and a single second law $F_1(\rho) \geq F_1(\sigma)$ suffices. Finite-copy thermodynamics is genuinely richer than the asymptotic second law.

Catalytic thermal operations

Allowing a "catalyst" — an auxiliary system that is returned unchanged — expands the set of reachable states. The classical result says catalysts do not give new constraints beyond thermomajorization, but they can smooth out the transformation. This is analogous to catalysts in classical chemistry.

Worked Example

Single-qubit thermomajorization

Take a qubit with $H = \frac{\omega}{2}(I - Z)$, so energies are 0 and ω. Gibbs populations at inverse temperature β:

$$\pi = (\pi_0, \pi_1) = \left(\frac{1}{1 + e^{-\beta\omega}},\ \frac{e^{-\beta\omega}}{1 + e^{-\beta\omega}}\right).$$

Consider two diagonal states: $\rho = (0.8, 0.2)$ and $\sigma = (0.7, 0.3)$.

Compute weights: $e^{\beta E_i} p_i$ for ρ: $(0.8, 0.2e^{\beta\omega})$. If $e^{\beta\omega} > 4$, then $0.2e^{\beta\omega} > 0.8$ and the ordering is (excited, ground). Otherwise (ground, excited).

Case $\beta\omega = 0$ (infinite temperature, Gibbs is maximally mixed):

Thermomajorization reduces to standard majorization. The curve for ρ plots $(0, 0) \to (0.5, 0.8) \to (1, 1)$; for σ, $(0, 0) \to (0.5, 0.7) \to (1, 1)$. ρ lies above σ, so $\rho \succ_\beta \sigma$ and $\rho \to \sigma$ is allowed.

Case $\beta\omega = 2$: Need to reorder by $e^{\beta E_i} p_i$. $e^{\beta\omega} p_1 = e^2 \cdot 0.2 \approx 1.48 > 0.8$, so we order (level 1, level 0). Curve for ρ uses x-coordinates $(0, e^{-\beta\omega}, 1) \approx (0, 0.135, 1)$. Compute and compare. At the intermediate point, ρ still lies above σ, so the transformation is allowed.

Case ρ below Gibbs, σ above Gibbs: Transformation is forbidden because thermal operations cannot increase the free energy.

Case both are Gibbs: Thermomajorization is trivially saturated in both directions; identity is the only transformation.

Work extraction as a resource process

A pure excited state $|1\rangle\langle 1|$ is an athermal resource. Extracting work from it means converting $|1\rangle$ to $|0\rangle$ while raising a work-storage system from $|w_0\rangle$ to $|w_1\rangle$, where $H_W|w_i\rangle = w_i|w_i\rangle$. The maximum extractable work is

$$W_{\max} = k_B T \log\left(\frac{1}{\pi_1}\right) = k_B T \log(1 + e^{\beta\omega}),$$

which for $\beta\omega \gg 1$ gives $W_{\max} \approx \omega$, and for $\beta\omega \ll 1$ gives $W_{\max} \approx k_B T \log 2$ (Landauer-like).

Cross-Links

- **9.1 Gibbs state** — the unique free state of thermal resource theory.
- **9.2 Work** — formalized as the athermality of a pure energy-eigenstate resource.
- **9.3 Fluctuation theorems** — consequences of monotone contractivity.
- **9.4 Landauer** — the $\alpha = 1$ monotone for the erasure transformation.
- **4.x Entanglement** — parallel resource theory; free = separable states, free = LOCC.
- **Coherence, magic, asymmetry** — further examples of the same structural template.
- **Information theory** — Rényi divergences are the core object; monotonicity under CPTP maps is the "data processing inequality."

Structural Compression

Invariant: Thermal operations define a partial order on states via thermomajorization, equivalently by preservation of all generalized free energies F_α.

Mechanism: Energy-conserving unitaries on system + Gibbs bath + partial trace = thermal operations. Monotones are Rényi divergences against Gibbs.

Cross-System Echo: Same template for entanglement (LOCC), coherence (incoherent ops), magic (Clifford), asymmetry (covariant ops). Quantum thermodynamics is one resource theory among several.

Exercises

1. Show that thermal operations map the Gibbs state to itself, and that this is the defining property of a "free state" in the resource-theory sense.
2. Derive the standard free energy $F_1(\rho) = \mathrm{tr}(\rho H) - TS(\rho)$ as the $\alpha \to 1$ limit of $F_\alpha(\rho) = \beta^{-1}[S_\alpha(\rho\|\rho_\beta) - \log Z]$ using L'Hôpital on the Rényi divergence.
3. For a qubit with energies $0, \omega$ at inverse temperature β, compute F_0, F_1, F_∞ for the pure excited state $|1\rangle\langle 1|$ and verify that all three are positive.
4. Show that two states with the same ordered eigenvalue list but different energy assignments have different thermomajorization curves, so the partial order really does depend on the Hamiltonian.
5. A thermal operation on a qubit cannot create coherence between energy eigenstates. Prove this using strict energy conservation of the Stinespring dilation.
6. Show that the Horodecki–Oppenheim thermomajorization criterion reduces to standard majorization (Hardy–Littlewood–Polya) in the infinite-temperature limit $\beta \to 0$. Interpret the result.

7. **Argue, structurally**, why a single second law is insufficient for single-shot quantum thermodynamics, and why exactly the Rényi family is the right generalization (rather than, say, arbitrary other convex functionals).

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Concepts: density-matrix, von-neumann-entropy, constrained-optimization
Prerequisites: 9.1

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