Events and Sigma-Algebras

Statistics · Module 1 — Probability Foundations

Node 1.2

Position in Knowledge Graph

System:
Statistics

Structural Domain:
Probability Foundations

Node:
1.2

Formal Definition

An **event** is a subset of the sample space Ω that we assign a probability to.

A **sigma-algebra** $\mathcal{F} \subseteq 2^{\Omega}$ is a collection of subsets of Ω such that:

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
3. If $A_1, A_2, \dots \in \mathcal{F}$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

From (2) and (3) it follows also that $\mathcal{F}$ is closed under countable intersections.

Intuition

A sigma-algebra is "what questions you are allowed to ask."

Sometimes, the full power set 2^{Ω} is too large or too pathological, so we restrict to a consistent family of events where probability behaves well.

Think:

- Ω: reality-space
- $\mathcal{F}$: the set of measurable questions
- $\mathbb{P}$: the answer-weighting function

Structural Role

Sigma-algebras are the "language layer" of probability.

They ensure:

- probabilities can be consistently assigned,
- limits of events remain events (countable operations),
- random variables can be defined rigorously (measurability depends on $\mathcal{F}$).

This is the gateway to random variables and integration (expectation).

Mathematical Development

Consequences

If $A_1, A_2, \dots \in \mathcal{F}$, then:

- Countable intersections:

  $$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$$

  because

  $$\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c \right)^c$$

  and sigma-algebras are closed under complements and countable unions.

- Finite unions and intersections are special cases of the countable rules.

Worked Examples

Example 1: Full power set (finite Ω)

If Ω is finite, then $\mathcal{F} = 2^{\Omega}$ is always a sigma-algebra.

Example 2: Trivial sigma-algebra

$$\mathcal{F} = \{\emptyset, \Omega\}$$

This corresponds to having only one meaningful question: "Did anything happen at all?"

Cross-Links

- **Statistics:** Models restrict which events are relevant (measurable) under a sampling process.
- **AI:** Feature representations define what distinctions the model can express (a "learned sigma-algebra" intuition).
- **Economics:** Information sets correspond to restricted event families (what an agent can distinguish).
- **Quantum:** Measurement choice defines the event structure (what outcomes are distinguishable).

Structural Compression

Invariant:
Closure under complement + countable union keeps probability consistent under limits.

Mechanism:
"Allowed questions" must be stable under repeated refinement.

Cross-System Echo:
In programming: closed interfaces; in physics: observable sets depend on measurement.

Exercises

1. Verify that $\{\emptyset, \Omega\}$ is a sigma-algebra.
2. If $\mathcal{F}$ is a sigma-algebra and $A, B \in \mathcal{F}$, prove that $A \setminus B \in \mathcal{F}$.
3. Let $\Omega = \{1, 2, 3, 4\}$. List all events in the sigma-algebra generated by the partition $\{\{1, 2\}, \{3, 4\}\}$.

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Concepts: probability-space

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