

Random Variables

Statistics · Module 1 — Probability Foundations

Node 1.3

Position in Knowledge Graph

System:

Statistics

Structural Domain:

Probability Foundations

Node:

1.3

Formal Definition

A **random variable** is a measurable function:

$$X : \Omega \rightarrow \mathbb{R}$$

such that for every Borel set $B \subseteq \mathbb{R}$,

$$X^{-1}(B) \in \mathcal{F}.$$

This measurability condition ensures that probabilities of events of the form $\{X \in B\}$ are well-defined.

Intuition

A random variable is not “a random number.”

It is a function that translates outcomes into numbers.

$$\text{Outcome} \longrightarrow \text{Number}$$

It allows us to measure, summarize, and analyze uncertainty quantitatively.

Structural Role

Random variables connect abstract probability spaces to numerical structure.

They enable:

- Expectation

- Variance
- Distribution functions
- Statistical modeling

They are the bridge from pure measure theory to statistics.

Mathematical Development

Induced Distribution

The random variable X induces a probability measure on $\mathbb{R}$:

$$\mathbb{P}_X(B) = \mathbb{P}(X \in B)$$

This is called the **pushforward measure** or distribution of X .

Discrete Example

If:

$$\Omega = \{H, T\}$$

Define:

$$X(H) = 1, \quad X(T) = 0.$$

Then:

$$\mathbb{P}(X = 1) = 0.5, \quad \mathbb{P}(X = 0) = 0.5.$$

Cross-Links

- **Linear Algebra:** Random vectors are vector-valued random variables.
 - **AI:** Data features are realizations of random variables.
 - **Economics:** Utility functions act on random states.
 - **Quantum:** Observables are operator-valued analogues of random variables.
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Structural Compression

Invariant:

Random variables preserve measurable structure.

Mechanism:

Function pushes probability measure forward.

Cross-System Echo:

In ML, feature maps transform input space into numeric representation.

Exercises

1. Prove that if X is measurable, then $aX + b$ is measurable.
2. Construct a random variable on a die roll representing parity.
3. Show how a CDF can be defined from the induced distribution.

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Concepts: random-variables, probability-space

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