Expectation

Statistics · Module 1 — Probability Foundations

Node 1.5

Position in Knowledge Graph

System:
Statistics

Structural Domain:
Probability Foundations

Node:
1.5

Formal Definition

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The **expectation** of X is defined as:

Discrete Case

$$\mathbb{E}[X] = \sum_x x\,\mathbb{P}(X = x),$$

provided the sum converges absolutely.

Continuous Case

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx,$$

provided the integral is finite.

General (Measure-Theoretic Form)

$$\mathbb{E}[X] = \int_\Omega X(\omega)\,d\mathbb{P}(\omega).$$

Intuition

Expectation is not "what usually happens."

It is the **probability-weighted average** of all possible outcomes.

It is the center of mass of the distribution.

If probability is mass, then expectation is balance.

Structural Role

Expectation is the fundamental linear operator of probability theory.

It enables:

- Definition of variance
- Risk evaluation in estimation
- Decision theory
- Optimization of statistical procedures
- Law of Large Numbers

Almost all statistical reasoning is built on properties of expectation.

Mathematical Development

Linearity

For random variables X, Y and constants a, b:

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y].$$

Linearity holds without independence.

Indicator Trick

For an event A, define:

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

Then:

$$\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A).$$

This connects expectation directly to probability.

Worked Example

Fair Die

$$\mathbb{E}[X] = \frac{1+2+3+4+5+6}{6} = 3.5.$$

Note: 3.5 is not an achievable outcome. Expectation need not lie in the support of X.

Cross-Links

- **Linear Algebra:** Expectation is a linear functional.
- **AI:** Loss functions are expectations over data distributions.
- **Economics:** Expected utility theory.
- **Quantum:** Expected value of an observable corresponds to $\langle\psi|A|\psi\rangle$.

Structural Compression

Invariant:
Expectation is linear.

Mechanism:
Integration against probability measure.

Cross-System Echo:
In physics, expectation resembles averaging over states; in ML, empirical averages approximate expectations.

Exercises

1. Prove linearity of expectation.
2. Show that $\mathbb{E}[c] = c$ for constant c.
3. Let $X \sim \text{Bernoulli}(p)$. Compute $\mathbb{E}[X]$.
4. Show that expectation of an indicator recovers probability.

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Concepts: expectation-value, random-variables

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