

Law of Large Numbers

Statistics · Module 1 — Probability Foundations

Node 1.8

Position in Knowledge Graph

System:

Statistics

Structural Domain:

Probability Foundations

Node:

1.8

Formal Statement

Let $X_1, X_2, \dots$ be independent and identically distributed (i.i.d.) random variables with finite expectation $\mu = \mathbb{E}[X_i]$.

Define the sample mean:

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

Weak Law of Large Numbers (WLLN)

$$\bar{X}_n \xrightarrow{P} \mu.$$

That is, the sample mean converges in probability to the true expectation.

Strong Law of Large Numbers (SLLN)

$$\bar{X}_n \xrightarrow{a.s.} \mu.$$

The convergence occurs almost surely.

Intuition

Averages stabilize.

As we collect more data, random fluctuations cancel out.

The sample mean becomes a reliable estimator of the true mean.

Probability turns into empirical regularity.

Structural Role

The Law of Large Numbers justifies:

- Statistical estimation
- Empirical averages
- Risk minimization
- Machine learning training procedures

Without LLN, statistics would not work.

Mathematical Insight

Variance of the sample mean:

$$\text{Var}(\bar{X}_n) = \frac{\text{Var}(X)}{n}.$$

As $n \rightarrow \infty$, variance shrinks to zero.

Concentration drives convergence.

Worked Example

Flip a fair coin.

Let $X_i = 1$ for heads, 0 for tails.

Then:

$$\bar{X}_n \rightarrow 0.5.$$

The empirical frequency converges to true probability.

Cross-Links

- **AI:** Empirical risk converges to expected risk.
 - **Economics:** Large populations stabilize aggregate behavior.
 - **Quantum:** Repeated measurements converge to expectation values.
 - **Linear Algebra:** Averaging as projection onto constant vector.
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Structural Compression

Invariant:

Sample mean approaches expectation.

Mechanism:

Variance shrinks as $1/n$.

Cross-System Echo:

In ML, more data reduces estimation noise.

Exercises

1. Prove convergence in probability using Chebyshev's inequality.
2. Compute variance of $\tilde{X}_n$.
3. Explain difference between weak and strong LLN.

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Concepts: law-of-large-numbers, convergence

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