Central Limit Theorem

Statistics · Module 1 — Probability Foundations

Node 1.9

Position in Knowledge Graph

System:
Statistics

Structural Domain:
Probability Foundations

Node:
1.9

Formal Statement

Let $X_1, X_2, \ldots$ be i.i.d. random variables with:

$$\mathbb{E}[X_i] = \mu, \quad \mathrm{Var}(X_i) = \sigma^2 < \infty.$$

Define:

$$Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}.$$

Then:

$$Z_n \xrightarrow{d} \mathcal{N}(0, 1).$$

Intuition

No matter the original distribution (under mild conditions), normalized averages become approximately normal.

The Gaussian distribution emerges universally.

Noise aggregates into structure.

Structural Role

The CLT enables:

- Approximate confidence intervals
- Hypothesis testing
- Asymptotic inference
- Statistical modeling at scale

It connects finite samples to Gaussian structure.

Mathematical Insight

The CLT explains why:

$$\bar{X}_n \approx \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

for large n.

This approximation powers classical statistics.

Worked Example

Let $X_i \sim \text{Bernoulli}(p)$.

Then:

$$\sqrt{n}(\bar{X}_n - p)$$

converges to:

$$\mathcal{N}(0, p(1-p)).$$

Cross-Links

- **AI:** SGD noise approximates Gaussian under large batch regimes.
- **Economics:** Aggregate shocks often modeled as Gaussian.
- **Quantum:** Gaussian approximations in large systems.
- **Linear Algebra:** Multivariate CLT yields multivariate normal with covariance matrix.

Structural Compression

Invariant:
Normal distribution emerges under aggregation.

Mechanism:
Independent fluctuations accumulate additively.

Cross-System Echo:
Universal Gaussian noise in physics and learning systems.

Exercises

1. State Lindeberg condition for CLT.
2. Show how CLT justifies normal approximation to binomial.
3. Explain difference between LLN and CLT.

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Concepts: central-limit-theorem, convergence

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