Confidence Sets

Statistics · Module 4 — Interval Estimation

Node 4.1

Position in Knowledge Graph

System:
Statistics

Structural Domain:
Interval Estimation

Node:
4.1

Statistical Setting

Let

$$X = (X_1, \ldots, X_n) \sim P_\theta,$$

where:

- $\theta \in \Theta \subseteq \mathbb{R}^k$,
- $\{P_\theta : \theta \in \Theta\}$ is a statistical model,
- inference is based on the observed data X.

In Domain 3, an estimator was a measurable function

$$\hat{\theta} : \mathcal{X} \to \Theta.$$

Interval estimation replaces point-valued estimators by set-valued estimators.

Definition — Confidence Set

A **confidence set** with level $1 - \alpha$, $0 < \alpha < 1$, is a measurable mapping

$$C : \mathcal{X} \to 2^\Theta$$

such that for every $\theta \in \Theta$,

$$\mathbb{P}_\theta\big(\theta \in C(X)\big) = 1 - \alpha.$$

Measurability requires that for each fixed θ,

$$\{X : \theta \in C(X)\}$$

is a measurable event under the sampling σ-algebra.

The probability is taken with respect to P_θ.
The parameter θ is fixed; randomness arises from the data.

Coverage Probability

For each $\theta \in \Theta$, define

$$\gamma(\theta) = \mathbb{P}_\theta\big(\theta \in C(X)\big).$$

Exact confidence requires

$$\gamma(\theta) = 1 - \alpha \quad \forall\theta \in \Theta.$$

If instead

$$\gamma(\theta) \geq 1 - \alpha \quad \forall\theta \in \Theta,$$

the procedure is conservative.

Finite-Sample and Asymptotic Confidence

Finite-Sample Validity

$$\mathbb{P}_\theta\big(\theta \in C(X)\big) = 1 - \alpha$$

holds exactly for each n.

Asymptotic Validity

A sequence $C_n(X)$ satisfies

$$\lim_{n\to\infty} \mathbb{P}_\theta\big(\theta \in C_n(X)\big) = 1 - \alpha.$$

Asymptotic confidence relies on limit distribution theory (LLN, CLT, Slutsky, delta method).

One-Dimensional Case

If $\theta \in \mathbb{R}$, a confidence set reduces to an interval

$$C(X) = [L(X), U(X)],$$

with

$$\mathbb{P}_\theta\big(L(X) \le \theta \le U(X)\big) = 1 - \alpha.$$

Here $L(X)$ and $U(X)$ are statistics.

Structural Role

Point estimation provides a single estimate.
Confidence sets quantify uncertainty around the parameter.

They formalize long-run coverage control and connect directly to hypothesis testing via duality (developed in Module 5).

Structural Compression

Invariant:
$\mathbb{P}_\theta\big(\theta \in C(X)\big) = 1 - \alpha.$

Mechanism:
Construct a random subset of parameter space whose sampling distribution controls coverage.

Cross-System Echo:
Confidence sets generalize error control: risk bounds in estimation and Type I error control in testing are parallel structures.

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Concepts: random-variables
Prerequisites: 1.3, 2.1, 3.1
Enables: 4.2, 4.3, 5.1

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