Pivotal Quantities

Statistics · Module 4 — Interval Estimation

Node 4.2

Statistical Setting

Let

$$X = (X_1, \dots, X_n) \sim P_\theta,$$

with parameter $\theta \in \Theta \subseteq \mathbb{R}^k$.

In Node 4.1, confidence sets were defined abstractly via coverage probability.
We now construct such sets using **pivotal quantities**.

Definition — Pivot

A statistic

$$T(X, \theta)$$

is called a **pivotal quantity** if its sampling distribution does not depend on the parameter θ.

Formally,

$$\mathcal{L}_\theta\big(T(X, \theta)\big)$$

is the same for all $\theta \in \Theta$.

Construction Principle

Suppose:

1. $T(X, \theta)$ is a pivot.
2. Its distribution function F_T is known.
3. Constants a_α, b_α satisfy

$$\mathbb{P}(a_\alpha \le T \le b_\alpha) = 1 - \alpha.$$

Then:

$$a_\alpha \le T(X, \theta) \le b_\alpha$$

can be algebraically inverted to produce a confidence set for θ.

Coverage follows because the distribution of T does not depend on θ.

Example — Normal Mean (Known Variance)

Let

$$X_i \sim \mathcal{N}(\mu, \sigma^2), \quad \sigma^2 \text{ known.}$$

Define

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}.$$

Then

$$Z \sim \mathcal{N}(0,1),$$

independently of μ.

Thus Z is a pivot.

Since

$$\mathbb{P}\big(-z_{\alpha/2} \le Z \le z_{\alpha/2}\big) = 1 - \alpha,$$

we obtain

$$\mathbb{P}\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

This yields the classical confidence interval.

Finite-Sample Exactness

When a true pivot exists and its distribution is known exactly, the resulting confidence set has finite-sample coverage:

$$\mathbb{P}_\theta(\theta \in C(X)) = 1 - \alpha.$$

Exact pivots typically arise in:

- Normal models
- Chi-square and t-distributions
- Exponential family models with known variance structure

Structural Role

Pivots provide the **finite-sample engine** of confidence theory.

They convert:

$$\text{Distributional identity}$$

into

$$\text{Coverage control.}$$

They are the constructive counterpart to the abstract definition in 4.1.

Structural Compression

Invariant

Distribution of $T(X, \theta)$ does not depend on θ.

Mechanism

Probability statement for pivot → algebraic inversion → confidence set.

Cross-System Echo

Symmetry removal: parameter dependence is normalized away, leaving a fixed reference distribution.

Statistics · Module 4 — Interval Estimation · Node 4.2
Concepts: random-variables, central-limit-theorem
Prerequisites: 4.1, 1.9, 3.2
Enables: 4.3, 5.1

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