Likelihood-Based Confidence Sets

Statistics · Module 4 — Interval Estimation

Node 4.3

Statistical Setting

Let

$$X \sim P_\theta, \quad \theta \in \Theta \subseteq \mathbb{R}^k,$$

with likelihood function

$$L(\theta) = L(\theta; X).$$

In Node 4.2, confidence sets were constructed from pivots.
We now construct them directly from the likelihood function.

Likelihood Ratio Statistic

Define the likelihood ratio:

$$\Lambda(\theta) = \frac{L(\theta)}{L(\hat{\theta})},$$

where $\hat{\theta}$ is the MLE.

Equivalently, in log-scale,

$$2 \log \Lambda(\theta) = 2\big(\ell(\theta) - \ell(\hat{\theta})\big).$$

Since $\ell(\hat{\theta}) \geq \ell(\theta)$, the statistic is nonpositive.

Define instead

$$W(\theta) = 2\big(\ell(\hat{\theta}) - \ell(\theta)\big).$$

Likelihood-Based Confidence Set

For a scalar parameter, define

$$C(X) = \{\theta \in \Theta : W(\theta) \leq c_\alpha\},$$

where c_α is chosen such that

$$\mathbb{P}_\theta\big(W(\theta) \leq c_\alpha\big) \approx 1 - \alpha.$$

Under regularity conditions and for large n,

$$W(\theta) \xrightarrow{d} \chi^2_1.$$

Thus one may choose

$$c_\alpha = \chi^2_{1,\,1-\alpha}.$$

Interpretation

Likelihood-based confidence sets consist of parameter values whose log-likelihood is sufficiently close to the maximum.

They are defined by:

$$\ell(\theta) \geq \ell(\hat{\theta}) - \frac{1}{2}c_\alpha.$$

Geometrically, they are level sets of the likelihood function.

Finite vs Asymptotic Validity

Unlike pivots, likelihood-based intervals are typically:

- Asymptotically valid,
- Not exactly finite-sample valid (except in special cases).

Their justification relies on:

- Asymptotic normality of MLE,
- Quadratic approximation of log-likelihood,
- Fisher information curvature.

These results are formalized in Domain 7.

Structural Role

Likelihood-based intervals:

- Are invariant under reparameterization,
- Extend naturally to multiparameter settings,
- Connect directly to likelihood ratio tests (Module 5),
- Reflect local information geometry.

They generalize pivot constructions when exact pivots do not exist.

Structural Compression

Invariant

Confidence set defined by likelihood level:

$$\ell(\theta) \geq \ell(\hat{\theta}) - \frac{1}{2}c_\alpha.$$

Mechanism

Likelihood curvature + asymptotic χ^2 distribution of LR statistic.

Cross-System Echo

Level-set geometry parallels quadratic approximation in optimization and second-order Taylor expansions.

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Concepts: likelihood, maximum-likelihood-estimation
Prerequisites: 4.1, 2.3, 2.4
Enables: 5.2, 7.2

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