Wald, Score, and Likelihood Ratio Intervals

Statistics · Module 4 — Interval Estimation

Node 4.4

Statistical Setting

Let

$$X_1, \ldots, X_n \sim P_\theta, \quad \theta \in \Theta \subseteq \mathbb{R}.$$

Assume:

1. The model satisfies regularity conditions for MLE asymptotics.
2. The MLE $\hat{\theta}_n$ exists and is consistent.
3. Fisher information $I(\theta)$ is positive and finite.
4. Asymptotic normality holds:

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}\big(0, I(\theta)^{-1}\big).$$

All asymptotic statements are in distribution under P_θ.

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1. Wald Interval

From asymptotic normality,

$$\frac{\hat{\theta}_n - \theta}{\sqrt{I(\hat{\theta}_n)^{-1}/n}} \xrightarrow{d} \mathcal{N}(0,1).$$

Thus,

$$C_n^W = \left[\hat{\theta}_n \pm z_{\alpha/2}\sqrt{\frac{1}{nI(\hat{\theta}_n)}}\right].$$

Coverage

$$\lim_{n\to\infty} \mathbb{P}_\theta(\theta \in C_n^W) = 1 - \alpha.$$

Wald intervals depend directly on the estimator scale.

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2. Score (Rao) Interval

Let

$$U_n(\theta) = \frac{\partial}{\partial\theta}\ell_n(\theta)$$

be the score function, and

$$I_n(\theta) = -\frac{\partial^2}{\partial\theta^2}\ell_n(\theta).$$

Under regularity,

$$\frac{U_n(\theta)}{\sqrt{I_n(\theta)}} \xrightarrow{d} \mathcal{N}(0,1).$$

The score confidence set is

$$C_n^S = \left\{\theta : \frac{U_n(\theta)^2}{I_n(\theta)} \le z_{\alpha/2}^2\right\}.$$

Coverage

$$\lim_{n\to\infty} \mathbb{P}_\theta(\theta \in C_n^S) = 1-\alpha.$$

Score intervals evaluate curvature at the candidate parameter, not at the estimator.

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3. Likelihood Ratio (LR) Interval

Define the likelihood ratio statistic:

$$\Lambda_n(\theta) = 2\big(\ell_n(\hat\theta_n) - \ell_n(\theta)\big).$$

Under standard regularity conditions,

$$\Lambda_n(\theta) \xrightarrow{d} \chi_1^2.$$

The LR confidence set is

$$C_n^{LR} = \left\{\theta : \Lambda_n(\theta) \le \chi_{1,1-\alpha}^2\right\}.$$

Coverage

$$\lim_{n\to\infty} \mathbb{P}_\theta(\theta \in C_n^{LR}) = 1-\alpha.$$

LR intervals are invariant under smooth reparameterization.

Comparative Structure

Method	Center	Information Evaluated At	Invariance
Wald	$\hat{\theta}$	Estimated parameter	Not invariant
Score	Solution set	Candidate θ	Partially invariant
LR	MLE	Global likelihood	Fully invariant

All three coincide to first order in $n^{-1/2}$, but differ in finite samples.

Structural Role

These are the three canonical asymptotic constructions:

- Wald: estimator-based geometry.
- Score: local curvature testing geometry.
- LR: global likelihood geometry.

They form the asymptotic backbone of modern inference and connect directly to hypothesis testing in Domain 5.

Structural Compression

Invariant

Asymptotic reference distribution (Normal or Chi-square) independent of θ.

Mechanism

Approximate pivotal statistic → inequality inversion → confidence set.

Cross-System Echo

Local quadratic expansion of log-likelihood (LAN structure) governs both interval estimation and hypothesis testing.

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Concepts: fisher-information, likelihood, central-limit-theorem
Prerequisites: 4.3, 2.4, 2.5, 7.2
Enables: 4.5, 5.3

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