Asymptotic Confidence Theory

Statistics · Module 4 — Interval Estimation

Node 4.5

Statistical Setting

Let

$$X_1, \ldots, X_n \sim P_\theta, \quad \theta \in \Theta \subseteq \mathbb{R}^k.$$

Let $C_n(X)$ be a sequence of measurable confidence sets.

All probability statements are taken under P_θ for fixed θ.

Definition — Asymptotic Level

The sequence C_n has **asymptotic level** $1 - \alpha$ if for every fixed $\theta \in \Theta$,

$$\lim_{n \to \infty} \mathbb{P}_\theta\big(\theta \in C_n(X)\big) = 1 - \alpha.$$

This is pointwise convergence in θ.

No uniformity over Θ is implied.

Uniform Asymptotic Coverage

If

$$\sup_{\theta \in \Theta} |\mathbb{P}_\theta(\theta \in C_n(X)) - (1 - \alpha)| \to 0,$$

then coverage holds uniformly over Θ.

Uniformity is strictly stronger than pointwise convergence.

Construction via Asymptotic Pivots

Suppose there exists a statistic $T_n(X, \theta)$ such that

$$T_n(X, \theta) \xrightarrow{d} T,$$

where the limiting distribution of T does not depend on θ.

Let constants a_α, b_α satisfy

$$\mathbb{P}(a_\alpha \le T \le b_\alpha) = 1 - \alpha.$$

Define

$$C_n(X) = \{\theta : a_\alpha \le T_n(X, \theta) \le b_\alpha\}.$$

If convergence in distribution holds under P_θ, then

$$\lim_{n\to\infty} \mathbb{P}_\theta(\theta \in C_n(X)) = 1 - \alpha.$$

This follows from the Portmanteau theorem when boundary probabilities vanish.

First-Order Expansion

In regular parametric models with scalar parameter,

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, I(\theta)^{-1}).$$

Replacing $I(\theta)$ by a consistent estimator yields

$$\frac{\hat{\theta}_n - \theta}{\sqrt{I(\hat{\theta}_n)^{-1}/n}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Intervals constructed from this expansion achieve asymptotic level $1 - \alpha$.

Higher-Order Considerations

Asymptotic coverage error can be expressed as

$$\mathbb{P}_\theta(\theta \in C_n) = 1 - \alpha + O(n^{-1/2}).$$

Higher-order corrections (e.g., Bartlett adjustments, Edgeworth expansions) reduce coverage error to smaller orders.

Such refinements require smoothness conditions and higher-order moment existence.

Structural Role

Finite-sample pivots rarely exist outside special models.

Asymptotic confidence theory generalizes interval construction by replacing exact distributional identities with limit laws.

It transfers the geometry of asymptotic normality into coverage control.

Structural Compression

Invariant

Limiting distribution independent of θ.

Mechanism

Asymptotic pivot → probability bound in limit → inverted inequality.

Cross-System Echo

Large-sample normality underlies both interval estimation and hypothesis testing; confidence level and test size share the same asymptotic reference law.

Statistics · Module 4 — Interval Estimation · Node 4.5
Concepts: convergence, central-limit-theorem
Prerequisites: 4.1, 4.4, 7.2, 7.4
Enables: 4.6, 5.6

hbar.university — own.your.cognition