Positive and Negative Feedback

Systems Thinking · Module 1 — Feedback Loops

Node 1.2

Position in Knowledge Graph

This node refines the general feedback mechanism (1.1) by classifying feedback according to its local effect on deviations from equilibrium.

It prepares for: - Oscillations and delays (1.5) - Fixed points (2.3) - Stability analysis (3.1)

Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}$ and consider a discrete-time feedback system

$$x_{t+1} = F(x_t), \quad F : \mathcal{X} \to \mathcal{X}.$$

Let $x^* \in \mathcal{X}$ be a fixed point, i.e.

$$F(x^*) = x^*.$$

Define the deviation variable

$$\delta_t = x_t - x^*.$$

Assume F is differentiable at x^*. A first-order Taylor expansion gives

$$F(x_t) = F(x^*) + F'(x^*)(x_t - x^*) + r(x_t),$$

where

$$\lim_{x_t \to x^*} \frac{r(x_t)}{x_t - x^*} = 0.$$

Using $F(x^*) = x^*$, we obtain

$$\delta_{t+1} = F'(x^*)\,\delta_t + r(x_t).$$

Neglecting higher-order terms locally:

$$\delta_{t+1} \approx F'(x^*)\,\delta_t.$$

Definition (Local Positive Feedback)

The system exhibits **local positive feedback** at x^* if

$$|F'(x^*)| > 1.$$

Small deviations are amplified:

$$|\delta_{t+1}| > |\delta_t| \quad \text{for sufficiently small } \delta_t \neq 0.$$

Definition (Local Negative Feedback)

The system exhibits **local negative feedback** at x^* if

$$|F'(x^*)| < 1.$$

Small deviations are attenuated:

$$|\delta_{t+1}| < |\delta_t| \quad \text{for sufficiently small } \delta_t \neq 0.$$

Assumptions and Conditions

1. $\mathcal{X} \subseteq \mathbb{R}$.
2. F is differentiable in a neighborhood of x^*.
3. The analysis is local.
4. Higher-order terms are neglected only for sufficiently small deviations.

This classification is local and does not imply global stability.

Proof or Mathematical Development

From the linearized deviation equation:

$$\delta_{t+1} \approx F'(x^*)\, \delta_t.$$

Iterating,

$$\delta_t \approx \left(F'(x^*)\right)^t \delta_0.$$

Case 1: $|F'(x^*)| < 1$.

Then

$$\lim_{t\to\infty} \left(F'(x^*)\right)^t = 0,$$

so

$$\delta_t \to 0.$$

Perturbations decay.

Case 2: $|F'(x^*)| > 1$.

Then

$$\left|(F'(x^*))^t\right| \to \infty,$$

so deviations grow.

Thus amplification vs attenuation is governed by the local gain $F'(x^*)$.

Structural Role

Positive and negative feedback determine:

- Divergence vs convergence
- Growth vs regulation
- Instability vs stabilization

All stability theory in later domains is a formalization of this local gain condition.

Positive feedback generates amplification and potential runaway behavior.

Negative feedback generates regulation and homeostasis.

This classification is the first quantitative refinement of the abstract feedback mechanism.

Structural Compression

Invariant

Local behavior near equilibrium is governed by the derivative:

$$\delta_{t+1} \approx F'(x^*)\,\delta_t.$$

Mechanism

Linearization and iteration of the feedback map determine amplification or attenuation.

Cross-System Echo

- Linear Algebra: spectral radius governs iteration of linear operators.
- Control Theory: loop gain determines closed-loop stability.
- Economics: expectation amplification vs stabilizing policy feedback.
- Biology: regulatory inhibition vs activation loops.

Systems Thinking · Module 1 — Feedback Loops · Node 1.2
Concepts: feedback, stability, fixed-point
Prerequisites: 1.1
Enables: 1.5, 2.3, 3.1

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