Stock-and-Flow Models

Systems Thinking · Module 1 — Feedback Loops

Node 1.4

Position in Knowledge Graph

This node formalizes stock-and-flow structure as a quantitative refinement of causal loop diagrams (1.3).

It builds on: - Feedback mechanisms (1.1) - Causal interaction structure (1.3)

It prepares for: - Time delays and oscillations (1.5) - Formal dynamical systems (2.1)

Stock-and-flow models introduce conservation structure and accumulation.

Formal Statement / Definition / Construction

Let $S = \{s_1, \ldots, s_n\}$ be a finite set of **stocks**.

A **stock** is a state variable

$$s_i : \mathbb{R}_{\geq 0} \to \mathbb{R},$$

interpreted as an accumulated quantity.

Let $F = \{f_1, \ldots, f_m\}$ be a finite set of **flows**.

Each flow

$$f_k : \mathbb{R}^n \to \mathbb{R}$$

represents a rate of change.

A **stock-and-flow model** is defined by differential equations of the form

$$\dot{s}_i(t) = \sum_{k \in \mathcal{I}_i} f_k(s(t)) - \sum_{k \in \mathcal{O}_i} f_k(s(t)),$$

where:

- $\mathcal{I}_i \subseteq \{1, \ldots, m\}$ indexes inflows to stock s_i,
- $\mathcal{O}_i \subseteq \{1, \ldots, m\}$ indexes outflows from stock s_i,
- $s(t) = (s_1(t), \ldots, s_n(t))$.

Equivalently, in vector form:

$$\dot{s}(t) = Bf(s(t)),$$

where:

- $f(s) \in \mathbb{R}^m$ is the flow vector,
- $B \in \mathbb{R}^{n \times m}$ is the incidence matrix, with entries in $\{-1, 0, +1\}$, encoding inflow $(+1)$ and outflow (-1) structure.

Assumptions and Conditions

1. Stocks are differentiable functions of time.
2. Flow functions are well-defined on $\mathbb{R}^n$.
3. If existence and uniqueness of solutions are required, assume flows are locally Lipschitz.
4. Conservation structure is encoded in matrix B.

No assumption of linearity is imposed.

Proof or Mathematical Development

A stock accumulates net inflow:

$$s_i(t) = s_i(0) + \int_0^t \dot{s}_i(\tau)\, d\tau.$$

Substituting the flow definition:

$$s_i(t) = s_i(0) + \int_0^t \left(\sum_{k \in \mathcal{I}_i} f_k(s(\tau)) - \sum_{k \in \mathcal{O}_i} f_k(s(\tau)) \right) d\tau.$$

Thus, stocks are integrators of flows.

If the system is closed (no external inflow or outflow), then total stock

$$S_{\text{total}}(t) = \sum_{i=1}^n s_i(t)$$

satisfies

$$\frac{d}{dt} S_{\text{total}}(t) = \sum_{i=1}^n \dot{s}_i(t).$$

Using matrix form:

$$\frac{d}{dt} S_{\text{total}}(t) = \mathbf{1}^\top B f(s(t)).$$

If

$$\mathbf{1}^\top B = 0,$$

then

$$\frac{d}{dt} S_{\text{total}}(t) = 0,$$

so total stock is conserved.

Thus conservation laws are linear algebraic constraints on B.

Structural Role

Stock-and-flow models introduce:

- Accumulation
- Conservation structure
- Integration over time
- Physical interpretability

They refine causal loop diagrams by distinguishing:

- Level variables (stocks)
- Rate variables (flows)

All continuous-time dynamical systems can be expressed in stock-and-flow form.

They are the bridge between qualitative feedback diagrams and quantitative differential equations.

Structural Compression

Invariant

Stocks evolve by integration of net flows:

$$\dot{s} = Bf(s).$$

Mechanism

Incidence matrix B encodes conservation and flow topology.

Cross-System Echo

- Linear Algebra: conservation corresponds to null space of $\mathbf{1}^\top B$.
- Physics: mass and energy balance equations.
- Economics: capital accumulation models.
- Biology: population dynamics with birth–death flows.

Systems Thinking · Module 1 — Feedback Loops · Node 1.4
Concepts: feedback, dynamical-systems, state-space
Prerequisites: 1.1, 1.3
Enables: 1.5, 2.1

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