Module 2 — Dynamical Systems

Systems Thinking

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State Spaces

Position in Knowledge Graph

This node formalizes the state space underlying feedback and stock–flow systems.

It builds on: - Feedback mechanisms (1.1) - Stock-and-flow structure (1.4)

It prepares for: - Phase portraits (2.2) - Fixed points (2.3) - Stability analysis (3.1)

State space provides the geometric container for trajectories.

Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}^n$.

A **state space** is a pair

$$(\mathcal{X}, \Phi),$$

where:

- $\mathcal{X}$ is a nonempty set (typically open or closed subset of $\mathbb{R}^n$),
- Φ is an evolution rule.

For discrete-time systems:

$$\Phi : \mathbb{N} \times \mathcal{X} \to \mathcal{X},$$

satisfying

$$\Phi(0, x) = x, \quad \Phi(t+1, x) = F(\Phi(t, x)),$$

for some update map

$$F : \mathcal{X} \to \mathcal{X}.$$

For continuous-time systems:

$$\Phi : \mathbb{R}_{\geq 0} \times \mathcal{X} \to \mathcal{X},$$

such that

$$\frac{d}{dt}\Phi(t, x) = f(\Phi(t, x)), \quad \Phi(0, x) = x,$$

for a vector field

$$f : \mathcal{X} \to \mathbb{R}^n.$$

The function $\Phi(t, x_0)$ is called the **trajectory** starting from initial state x_0.

Assumptions and Conditions

1. $\mathcal{X} \subseteq \mathbb{R}^n$.
2. For continuous-time systems, assume f is locally Lipschitz to ensure existence and uniqueness of solutions.
3. For discrete-time systems, F must be well-defined on $\mathcal{X}$.
4. Time is either discrete ($\mathbb{N}$) or continuous ($\mathbb{R}_{\geq 0}$).

No assumption of linearity, stability, or boundedness is imposed.

Proof or Mathematical Development

For discrete-time systems:

$$x_{t+1} = F(x_t).$$

Iterating:

$$x_t = F^t(x_0),$$

where

$$F^t = \underbrace{F \circ F \circ \cdots \circ F}_{t \text{ times}}.$$

Thus:

$$\Phi(t, x_0) = F^t(x_0).$$

For continuous-time systems:

$$\dot{x}(t) = f(x(t)).$$

Under local Lipschitz continuity, Picard–Lindelöf theorem ensures existence and uniqueness of solutions for given initial condition $x(0) = x_0$.

Thus the evolution operator satisfies:

$$\Phi(t, x_0) = x(t),$$

where $x(t)$ solves the differential equation.

In both cases, state space provides the domain over which trajectories evolve.

Structural Role

State space:

- Defines the set of possible system configurations.
- Encodes dimensionality.
- Enables geometric interpretation of dynamics.
- Serves as the domain for stability, bifurcation, and control analysis.

All later structural analysis operates inside $\mathcal{X}$.

Without a defined state space, trajectories and equilibria are undefined.

Structural Compression

Invariant

A dynamical system is evolution on a state space:

$$\Phi : T \times \mathcal{X} \to \mathcal{X}.$$

Mechanism

Iteration (discrete) or integration of a vector field (continuous) generates trajectories.

Cross-System Echo

- Linear Algebra: state vectors in $\mathbb{R}^n$.
- Control Theory: state representation of systems.
- Economics: capital and state-variable models.
- Physics: phase space of mechanical systems.

Phase Portraits

Position in Knowledge Graph

This node provides the geometric representation of trajectories inside a state space.

It builds on: - State spaces (2.1)

It prepares for: - Fixed points (2.3) - Limit cycles (2.5) - Stability theory (3.1)

Phase portraits translate evolution rules into geometric structure.

Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}^n$.

Consider a continuous-time dynamical system:

$$\dot{x}(t) = f(x(t)), \quad f : \mathcal{X} \to \mathbb{R}^n.$$

The **phase portrait** of the system is the collection:

$$\mathcal{P} = \left\{ \{\Phi(t, x_0) : t \in \mathbb{R}_{\geq 0}\} \ : \ x_0 \in \mathcal{X} \right\},$$

where $\Phi(t, x_0)$ is the trajectory starting from x_0.

In the case $n = 2$, the phase portrait is a planar vector field together with all integral curves satisfying:

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}, \quad f_1(x_1, x_2) \neq 0.$$

Assumptions and Conditions

1. $\mathcal{X} \subseteq \mathbb{R}^n$.
2. f is locally Lipschitz.
3. Trajectories are unique for given initial conditions.
4. The portrait describes forward-time evolution unless otherwise specified.

No assumption of linearity or stability is imposed.

Proof or Mathematical Development

For each $x_0 \in \mathcal{X}$, existence and uniqueness guarantee a unique solution:

$$x(t) = \Phi(t, x_0).$$

The phase portrait is therefore a partition of $\mathcal{X}$ into trajectory curves.

For planar systems:

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2),$$

we obtain:

$$\frac{dx_2}{dt} = f_2(x_1, x_2), \quad \frac{dx_1}{dt} = f_1(x_1, x_2).$$

By the chain rule:

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}.$$

Thus trajectories are integral curves of the slope field defined by the vector field f.

In linear systems:

$$\dot{x} = Ax,$$

solutions are:

$$x(t) = e^{At} x_0,$$

so the phase portrait is determined by eigenvalues and eigenvectors of A.

Structural Role

Phase portraits:

- Provide geometric visualization of dynamics.
- Identify invariant sets.
- Reveal qualitative behavior without solving explicitly.
- Enable classification of equilibria (node, saddle, spiral).

They transform algebraic evolution rules into geometric structure.

All stability and bifurcation analysis operates on phase portrait geometry.

Structural Compression

Invariant

The phase portrait is the set of all trajectories:

$$\mathcal{P} = \{\Phi(\cdot, x_0) : x_0 \in \mathcal{X}\}.$$

Mechanism

Vector field f determines integral curves via:

$$\dot{x} = f(x).$$

Cross-System Echo

- Linear Algebra: eigenstructure shapes trajectories.
- Physics: phase space diagrams in mechanics.

- Control Theory: state-space flow geometry.
- Economics: phase diagrams in growth models.

Fixed Points

Position in Knowledge Graph

This node defines equilibrium structure inside a state space.

It builds on: - State spaces (2.1) - Phase portraits (2.2)

It prepares for: - Bifurcations (2.4) - Stability theory (3.1)

Fixed points are the stationary anchors of trajectories.

Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}^n$.

Discrete-Time System

Given

$$x_{t+1} = F(x_t), \quad F : \mathcal{X} \to \mathcal{X},$$

a point $x^* \in \mathcal{X}$ is a **fixed point** if

$$F(x^*) = x^*.$$

Continuous-Time System

Given

$$\dot{x}(t) = f(x(t)), \quad f : \mathcal{X} \to \mathbb{R}^n,$$

a point $x^* \in \mathcal{X}$ is a **fixed point** (equilibrium point) if

$$f(x^*) = 0.$$

In this case, the constant trajectory

$$x(t) \equiv x^*$$

is a solution.

Assumptions and Conditions

1. $\mathcal{X} \subseteq \mathbb{R}^n$.
2. F is defined on $\mathcal{X}$.
3. For continuous systems, assume f is continuous to ensure equilibrium is well-defined.
4. Existence of fixed points is not guaranteed; it depends on system structure.

No stability assumption is imposed at this stage.

Proof or Mathematical Development

Discrete-Time Case

If x^* satisfies

$$F(x^*) = x^*,$$

then

$$x_1 = F(x^*) = x^*,$$

and inductively,

$$x_t = F^t(x^*) = x^* \quad \forall t \in \mathbb{N}.$$

Thus the trajectory remains constant.

Continuous-Time Case

If

$$f(x^*) = 0,$$

then

$$\dot{x}(t) = 0$$

when $x(t) = x^*$.

Thus the constant function $x(t) \equiv x^*$ satisfies the differential equation.

Linear Example

Consider

$$\dot{x} = Ax + b, \quad A \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^n.$$

A fixed point satisfies

$$Ax^* + b = 0.$$

If A is invertible,

$$x^* = -A^{-1}b.$$

Existence and uniqueness depend on invertibility of A.

Structural Role

Fixed points:

- Represent stationary system configurations.
- Serve as reference points for stability analysis.
- Anchor phase portrait structure.
- Provide the basis for linearization.

All local stability, oscillation, and bifurcation theory is defined relative to fixed points.

Structural Compression

Invariant

A fixed point satisfies:

$$F(x^*) = x^* \quad \text{or} \quad f(x^*) = 0.$$

Mechanism

Equilibrium occurs when update or velocity vanishes.

Cross-System Echo

- Linear Algebra: solutions of $Ax = 0$.
- Control Theory: equilibrium operating points.
- Economics: market-clearing equilibrium.
- Biology: steady-state population levels.

Bifurcations

Position in Knowledge Graph

This node introduces structural change in dynamical systems as parameters vary.

It builds on: - Fixed points (2.3)

It prepares for: - Limit cycles (2.5) - Stability theory (3.1)

Bifurcation theory studies qualitative changes in phase portrait structure.

Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}^n$.

Consider a parameterized dynamical system:

$$\dot{x} = f(x, \mu), \quad f : \mathcal{X} \times \mathbb{R} \to \mathbb{R}^n,$$

where $\mu \in \mathbb{R}$ is a scalar parameter.

A **bifurcation** occurs at $\mu = \mu^*$ if the qualitative structure of the phase portrait changes as μ passes through μ^*.

Formally, a bifurcation occurs when:

1. A fixed point changes stability, or
2. The number of fixed points changes, or
3. A new invariant set (e.g., periodic orbit) appears or disappears.

Assumptions and Conditions

1. f is continuously differentiable in x and μ.
2. Fixed points satisfy:
$$f(x^*, \mu) = 0.$$
3. Stability is determined by eigenvalues of the Jacobian:
$$J(x^*, \mu) = \frac{\partial f}{\partial x}(x^*, \mu).$$
4. A bifurcation requires a loss of hyperbolicity:
$$\det(J(x^*, \mu^*)) = 0 \quad \text{or} \quad \mathrm{Re}(\lambda_i(\mu^*)) = 0.$$

Hyperbolic equilibria (no eigenvalues with zero real part) are structurally stable under small perturbations.

Proof or Mathematical Development

Let $x^*(\mu)$ be a branch of fixed points satisfying:

$$f(x^*(\mu), \mu) = 0.$$

Linearizing near x^*:

$$\dot{y} = J(x^*, \mu)y.$$

Stability depends on eigenvalues $\lambda_i(\mu)$.

If for some $\mu = \mu^*$,

$$\mathrm{Re}(\lambda_i(\mu^*)) = 0,$$

then hyperbolicity fails.

Example: Saddle-node bifurcation.

Consider

$$\dot{x} = \mu - x^2.$$

Fixed points satisfy:

$$\mu - x^2 = 0 \quad \Rightarrow \quad x^* = \pm\sqrt{\mu}.$$

If $\mu < 0$, no real fixed points exist. If $\mu = 0$, one fixed point at $x = 0$. If $\mu > 0$, two fixed points.

Thus at $\mu = 0$, the number of equilibria changes.

Jacobian:

$$\frac{d}{dx}(\mu - x^2) = -2x.$$

At $x = 0$, eigenvalue $\lambda = 0$.

Loss of hyperbolicity marks the bifurcation point.

Structural Role

Bifurcations:

- Mark thresholds of qualitative change.
- Define tipping points.
- Explain regime shifts in biological, economic, and engineered systems.
- Connect local stability to global structure.

They formalize when feedback structure produces new behavior.

Structural Compression

Invariant

Bifurcation requires loss of hyperbolicity:

$$\mathrm{Re}(\lambda_i(\mu^*)) = 0.$$

Mechanism

Parameter variation alters Jacobian eigenvalues, changing phase portrait topology.

Cross-System Echo

- Control Theory: stability margins and critical gain.
- Economics: regime shifts and tipping equilibria.
- Physics: phase transitions.
- Biology: population collapse thresholds.

Limit Cycles

Position in Knowledge Graph

This node formalizes persistent oscillatory behavior in continuous-time dynamical systems.

It builds on: - Phase portraits (2.2) - Bifurcations (2.4)

It prepares for: - Lyapunov stability (3.1) - Pattern formation and emergence (5.3)

Limit cycles represent self-sustained oscillations.

Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}^2$.

Consider a continuous-time dynamical system:

$$\dot{x} = f(x), \quad f : \mathcal{X} \to \mathbb{R}^2,$$

with f continuously differentiable.

A **periodic orbit** is a nonconstant trajectory $\gamma(t)$ satisfying:

$$\gamma(t + T) = \gamma(t) \quad \forall t,$$

for some minimal period $T > 0$.

A **limit cycle** is an isolated periodic orbit $\Gamma \subset \mathcal{X}$ such that:

1. Γ is a closed trajectory.
2. There exists a neighborhood U of Γ such that trajectories in $U \setminus \Gamma$ approach Γ as $t \to +\infty$ (stable limit cycle), or as $t \to -\infty$ (unstable limit cycle).

Assumptions and Conditions

1. The system is two-dimensional.
2. f is continuously differentiable.
3. The periodic orbit is isolated in phase space.
4. Stability is defined with respect to nearby trajectories.

No assumption of linearity is imposed.

Proof or Mathematical Development

Example: Van der Pol-Type Oscillator (Qualitative Form)

Consider the planar system:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \mu(1 - x_1^2)x_2 - x_1, \quad \mu > 0.$$

For small amplitudes ($|x_1| < 1$):

$$\mu(1 - x_1^2) > 0,$$

so energy is injected into the system.

For large amplitudes ($|x_1| > 1$):

$$\mu(1 - x_1^2) < 0,$$

so energy is dissipated.

Thus:

- Small oscillations grow.
- Large oscillations shrink.

This creates a closed orbit that is attracting.

The origin may be unstable (depending on μ), but trajectories neither diverge to infinity nor converge to equilibrium. Instead, they approach a periodic orbit.

Isolation Property

Let Γ be a periodic orbit.

It is a limit cycle if no other periodic orbits exist in a neighborhood of Γ.

This distinguishes limit cycles from centers (e.g., linear harmonic oscillator), where infinitely many closed orbits exist.

Linear Systems Cannot Have Limit Cycles

Consider a linear system:

$$\dot{x} = Ax.$$

If eigenvalues are purely imaginary and nondefective, the system produces closed orbits, but they are not isolated.

Thus linear systems cannot produce limit cycles.

Limit cycles require nonlinearity.

Structural Role

Limit cycles represent:

- Sustained oscillation.
- Self-regulation through nonlinear feedback.
- Stable dynamic regimes distinct from equilibrium.

They are central to:

- Biological rhythms.
- Economic cycles.

- Electrical oscillators.
- Predator–prey dynamics.

They emerge when feedback and nonlinearity combine.

Structural Compression

Invariant

A limit cycle is an isolated periodic orbit:

$$\gamma(t+T) = \gamma(t), \quad \Gamma \text{ isolated.}$$

Mechanism

Nonlinear feedback injects and dissipates energy depending on amplitude.

Cross-System Echo

- Control Theory: oscillatory closed-loop behavior.
- Biology: circadian rhythms.
- Economics: endogenous business cycles.
- Physics: nonlinear oscillators.