# State Spaces

Systems Thinking · Module 2 — Dynamical Systems

Node 2.1

## Position in Knowledge Graph

This node formalizes the state space underlying feedback and stock–flow systems.

It builds on: - Feedback mechanisms (1.1) - Stock-and-flow structure (1.4)

It prepares for: - Phase portraits (2.2) - Fixed points (2.3) - Stability analysis (3.1)

State space provides the geometric container for trajectories.

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## Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}^n$.

A **state space** is a pair

$$(\mathcal{X}, \Phi),$$

where:

- $\mathcal{X}$ is a nonempty set (typically open or closed subset of $\mathbb{R}^n$),
- $\Phi$ is an evolution rule.

For discrete-time systems:

$$\Phi : \mathbb{N} \times \mathcal{X} \to \mathcal{X},$$

satisfying

$$\Phi(0, x) = x, \quad \Phi(t+1, x) = F(\Phi(t, x)),$$

for some update map

$$F : \mathcal{X} \to \mathcal{X}.$$

For continuous-time systems:

$$\Phi : \mathbb{R}_{\geq 0} \times \mathcal{X} \to \mathcal{X},$$

such that

$$\frac{d}{dt}\Phi(t, x) = f(\Phi(t, x)), \quad \Phi(0, x) = x,$$

for a vector field

$$f : \mathcal{X} \to \mathbb{R}^n.$$

The function $\Phi(t, x_0)$ is called the **trajectory** starting from initial state $x_0$.

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## Assumptions and Conditions

1. $\mathcal{X} \subseteq \mathbb{R}^n$.
2. For continuous-time systems, assume $f$ is locally Lipschitz to ensure existence and uniqueness of solutions.
3. For discrete-time systems, $F$ must be well-defined on $\mathcal{X}$.
4. Time is either discrete ($\mathbb{N}$) or continuous ($\mathbb{R}_{\geq 0}$).

No assumption of linearity, stability, or boundedness is imposed.

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## Proof or Mathematical Development

For discrete-time systems:

$$x_{t+1} = F(x_t).$$

Iterating:

$$x_t = F^t(x_0),$$

where

$$F^t = \underbrace{F \circ F \circ \cdots \circ F}_{t \text{ times}}.$$

Thus:

$$\Phi(t, x_0) = F^t(x_0).$$

For continuous-time systems:

$$\dot{x}(t) = f(x(t)).$$

Under local Lipschitz continuity, Picard–Lindelöf theorem ensures existence and uniqueness of solutions for given initial condition $x(0) = x_0$.

Thus the evolution operator satisfies:

$$\Phi(t, x_0) = x(t),$$

where $x(t)$ solves the differential equation.

In both cases, state space provides the domain over which trajectories evolve.

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## Structural Role

State space:

- Defines the set of possible system configurations.
- Encodes dimensionality.
- Enables geometric interpretation of dynamics.
- Serves as the domain for stability, bifurcation, and control analysis.

All later structural analysis operates inside $\mathcal{X}$.

Without a defined state space, trajectories and equilibria are undefined.

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## Structural Compression

### Invariant

A dynamical system is evolution on a state space:

$$\Phi : T \times \mathcal{X} \to \mathcal{X}.$$

### Mechanism

Iteration (discrete) or integration of a vector field (continuous) generates trajectories.

### Cross-System Echo

- Linear Algebra: state vectors in $\mathbb{R}^n$.
- Control Theory: state representation of systems.
- Economics: capital and state-variable models.
- Physics: phase space of mechanical systems.

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*Systems Thinking · Module 2 — Dynamical Systems · Node 2.1*
**Concepts:** state-space, dynamical-systems
**Prerequisites:** 1.1, 1.4
**Enables:** 2.2, 2.3, 3.1

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