Phase Portraits

Systems Thinking · Module 2 — Dynamical Systems

Node 2.2

Position in Knowledge Graph

This node provides the geometric representation of trajectories inside a state space.

It builds on: - State spaces (2.1)

It prepares for: - Fixed points (2.3) - Limit cycles (2.5) - Stability theory (3.1)

Phase portraits translate evolution rules into geometric structure.

Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}^n$.

Consider a continuous-time dynamical system:

$$\dot{x}(t) = f(x(t)), \quad f : \mathcal{X} \to \mathbb{R}^n.$$

The **phase portrait** of the system is the collection:

$$\mathcal{P} = \left\{ \{\Phi(t, x_0) : t \in \mathbb{R}_{\geq 0}\} \ : \ x_0 \in \mathcal{X} \right\},$$

where $\Phi(t, x_0)$ is the trajectory starting from x_0.

In the case $n = 2$, the phase portrait is a planar vector field together with all integral curves satisfying:

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}, \quad f_1(x_1, x_2) \neq 0.$$

Assumptions and Conditions

1. $\mathcal{X} \subseteq \mathbb{R}^n$.
2. f is locally Lipschitz.
3. Trajectories are unique for given initial conditions.
4. The portrait describes forward-time evolution unless otherwise specified.

No assumption of linearity or stability is imposed.

Proof or Mathematical Development

For each $x_0 \in \mathcal{X}$, existence and uniqueness guarantee a unique solution:

$$x(t) = \Phi(t, x_0).$$

The phase portrait is therefore a partition of $\mathcal{X}$ into trajectory curves.

For planar systems:

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2),$$

we obtain:

$$\frac{dx_2}{dt} = f_2(x_1, x_2), \quad \frac{dx_1}{dt} = f_1(x_1, x_2).$$

By the chain rule:

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}.$$

Thus trajectories are integral curves of the slope field defined by the vector field f.

In linear systems:

$$\dot{x} = Ax,$$

solutions are:

$$x(t) = e^{At}x_0,$$

so the phase portrait is determined by eigenvalues and eigenvectors of A.

Structural Role

Phase portraits:

- Provide geometric visualization of dynamics.
- Identify invariant sets.
- Reveal qualitative behavior without solving explicitly.
- Enable classification of equilibria (node, saddle, spiral).

They transform algebraic evolution rules into geometric structure.

All stability and bifurcation analysis operates on phase portrait geometry.

Structural Compression

Invariant

The phase portrait is the set of all trajectories:

$$\mathcal{P} = \{\Phi(\cdot, x_0) : x_0 \in \mathcal{X}\}.$$

Mechanism

Vector field f determines integral curves via:

$$\dot{x} = f(x).$$

Cross-System Echo

- Linear Algebra: eigenstructure shapes trajectories.
- Physics: phase space diagrams in mechanics.
- Control Theory: state-space flow geometry.
- Economics: phase diagrams in growth models.

Systems Thinking · Module 2 — Dynamical Systems · Node 2.2
Concepts: dynamical-systems, state-space, phase-portrait
Prerequisites: 2.1
Enables: 2.3, 2.5, 3.1

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