Fixed Points

Systems Thinking · Module 2 — Dynamical Systems

Node 2.3

Position in Knowledge Graph

This node defines equilibrium structure inside a state space.

It builds on: - State spaces (2.1) - Phase portraits (2.2)

It prepares for: - Bifurcations (2.4) - Stability theory (3.1)

Fixed points are the stationary anchors of trajectories.

Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}^n$.

Discrete-Time System

Given

$$x_{t+1} = F(x_t), \quad F : \mathcal{X} \to \mathcal{X},$$

a point $x^* \in \mathcal{X}$ is a **fixed point** if

$$F(x^*) = x^*.$$

Continuous-Time System

Given

$$\dot{x}(t) = f(x(t)), \quad f : \mathcal{X} \to \mathbb{R}^n,$$

a point $x^* \in \mathcal{X}$ is a **fixed point** (equilibrium point) if

$$f(x^*) = 0.$$

In this case, the constant trajectory

$$x(t) \equiv x^*$$

is a solution.

Assumptions and Conditions

1. $\mathcal{X} \subseteq \mathbb{R}^n$.
2. F is defined on $\mathcal{X}$.
3. For continuous systems, assume f is continuous to ensure equilibrium is well-defined.
4. Existence of fixed points is not guaranteed; it depends on system structure.

No stability assumption is imposed at this stage.

Proof or Mathematical Development

Discrete-Time Case

If x^* satisfies

$$F(x^*) = x^*,$$

then

$$x_1 = F(x^*) = x^*,$$

and inductively,

$$x_t = F^t(x^*) = x^* \quad \forall t \in \mathbb{N}.$$

Thus the trajectory remains constant.

Continuous-Time Case

If

$$f(x^*) = 0,$$

then

$$\dot{x}(t) = 0$$

when $x(t) = x^*$.

Thus the constant function $x(t) \equiv x^*$ satisfies the differential equation.

Linear Example

Consider

$$\dot{x} = Ax + b, \quad A \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^n.$$

A fixed point satisfies

$$Ax^* + b = 0.$$

If A is invertible,

$$x^* = -A^{-1}b.$$

Existence and uniqueness depend on invertibility of A.

Structural Role

Fixed points:

- Represent stationary system configurations.
- Serve as reference points for stability analysis.
- Anchor phase portrait structure.
- Provide the basis for linearization.

All local stability, oscillation, and bifurcation theory is defined relative to fixed points.

Structural Compression

Invariant

A fixed point satisfies:

$$F(x^*) = x^* \quad \text{or} \quad f(x^*) = 0.$$

Mechanism

Equilibrium occurs when update or velocity vanishes.

Cross-System Echo

- Linear Algebra: solutions of $Ax = 0$.
- Control Theory: equilibrium operating points.
- Economics: market-clearing equilibrium.
- Biology: steady-state population levels.

Systems Thinking · Module 2 — Dynamical Systems · Node 2.3
Concepts: dynamical-systems, fixed-point, stability
Prerequisites: 2.1, 2.2
Enables: 2.4, 3.1

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