Bifurcations

Systems Thinking · Module 2 — Dynamical Systems

Node 2.4

Position in Knowledge Graph

This node introduces structural change in dynamical systems as parameters vary.

It builds on: - Fixed points (2.3)

It prepares for: - Limit cycles (2.5) - Stability theory (3.1)

Bifurcation theory studies qualitative changes in phase portrait structure.

Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}^n$.

Consider a parameterized dynamical system:

$$\dot{x} = f(x, \mu), \quad f : \mathcal{X} \times \mathbb{R} \to \mathbb{R}^n,$$

where $\mu \in \mathbb{R}$ is a scalar parameter.

A **bifurcation** occurs at $\mu = \mu^*$ if the qualitative structure of the phase portrait changes as μ passes through μ^*.

Formally, a bifurcation occurs when:

1. A fixed point changes stability, or
2. The number of fixed points changes, or
3. A new invariant set (e.g., periodic orbit) appears or disappears.

Assumptions and Conditions

1. f is continuously differentiable in x and μ.
2. Fixed points satisfy:
$$f(x^*, \mu) = 0.$$
3. Stability is determined by eigenvalues of the Jacobian:
$$J(x^*, \mu) = \frac{\partial f}{\partial x}(x^*, \mu).$$
4. A bifurcation requires a loss of hyperbolicity:
$$\det(J(x^*, \mu^*)) = 0 \quad \text{or} \quad \mathrm{Re}(\lambda_i(\mu^*)) = 0.$$

Hyperbolic equilibria (no eigenvalues with zero real part) are structurally stable under small perturbations.

Proof or Mathematical Development

Let $x^*(\mu)$ be a branch of fixed points satisfying:

$$f(x^*(\mu), \mu) = 0.$$

Linearizing near x^*:

$$\dot{y} = J(x^*, \mu)y.$$

Stability depends on eigenvalues $\lambda_i(\mu)$.

If for some $\mu = \mu^*$,

$$\mathrm{Re}(\lambda_i(\mu^*)) = 0,$$

then hyperbolicity fails.

Example: Saddle-node bifurcation.

Consider

$$\dot{x} = \mu - x^2.$$

Fixed points satisfy:

$$\mu - x^2 = 0 \quad \Rightarrow \quad x^* = \pm\sqrt{\mu}.$$

If $\mu < 0$, no real fixed points exist. If $\mu = 0$, one fixed point at $x = 0$. If $\mu > 0$, two fixed points.

Thus at $\mu = 0$, the number of equilibria changes.

Jacobian:

$$\frac{d}{dx}(\mu - x^2) = -2x.$$

At $x = 0$, eigenvalue $\lambda = 0$.

Loss of hyperbolicity marks the bifurcation point.

Structural Role

Bifurcations:

- Mark thresholds of qualitative change.
- Define tipping points.
- Explain regime shifts in biological, economic, and engineered systems.
- Connect local stability to global structure.

They formalize when feedback structure produces new behavior.

Structural Compression

Invariant

Bifurcation requires loss of hyperbolicity:

$$\mathrm{Re}(\lambda_i(\mu^*)) = 0.$$

Mechanism

Parameter variation alters Jacobian eigenvalues, changing phase portrait topology.

Cross-System Echo

- Control Theory: stability margins and critical gain.
- Economics: regime shifts and tipping equilibria.
- Physics: phase transitions.
- Biology: population collapse thresholds.

Systems Thinking · Module 2 — Dynamical Systems · Node 2.4
Concepts: dynamical-systems, bifurcation, phase-portrait
Prerequisites: 2.3
Enables: 2.5, 3.1

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