Limit Cycles

Systems Thinking · Module 2 — Dynamical Systems

Node 2.5

Position in Knowledge Graph

This node formalizes persistent oscillatory behavior in continuous-time dynamical systems.

It builds on: - Phase portraits (2.2) - Bifurcations (2.4)

It prepares for: - Lyapunov stability (3.1) - Pattern formation and emergence (5.3)

Limit cycles represent self-sustained oscillations.

Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}^2$.

Consider a continuous-time dynamical system:

$$\dot{x} = f(x), \quad f : \mathcal{X} \to \mathbb{R}^2,$$

with f continuously differentiable.

A **periodic orbit** is a nonconstant trajectory $\gamma(t)$ satisfying:

$$\gamma(t + T) = \gamma(t) \quad \forall t,$$

for some minimal period $T > 0$.

A **limit cycle** is an isolated periodic orbit $\Gamma \subset \mathcal{X}$ such that:

1. Γ is a closed trajectory.
2. There exists a neighborhood U of Γ such that trajectories in $U \setminus \Gamma$ approach Γ as $t \to +\infty$ (stable limit cycle), or as $t \to -\infty$ (unstable limit cycle).

Assumptions and Conditions

1. The system is two-dimensional.
2. f is continuously differentiable.
3. The periodic orbit is isolated in phase space.
4. Stability is defined with respect to nearby trajectories.

No assumption of linearity is imposed.

Proof or Mathematical Development

Example: Van der Pol-Type Oscillator (Qualitative Form)

Consider the planar system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \mu(1 - x_1^2)x_2 - x_1, \quad \mu > 0. \end{aligned}$$

For small amplitudes ($|x_1| < 1$):

$$\mu(1 - x_1^2) > 0,$$

so energy is injected into the system.

For large amplitudes ($|x_1| > 1$):

$$\mu(1 - x_1^2) < 0,$$

so energy is dissipated.

Thus:

- Small oscillations grow.
- Large oscillations shrink.

This creates a closed orbit that is attracting.

The origin may be unstable (depending on μ), but trajectories neither diverge to infinity nor converge to equilibrium. Instead, they approach a periodic orbit.

Isolation Property

Let Γ be a periodic orbit.

It is a limit cycle if no other periodic orbits exist in a neighborhood of Γ.

This distinguishes limit cycles from centers (e.g., linear harmonic oscillator), where infinitely many closed orbits exist.

Linear Systems Cannot Have Limit Cycles

Consider a linear system:

$$\dot{x} = Ax.$$

If eigenvalues are purely imaginary and nondefective, the system produces closed orbits, but they are not isolated.

Thus linear systems cannot produce limit cycles.

Limit cycles require nonlinearity.

Structural Role

Limit cycles represent:

- Sustained oscillation.
- Self-regulation through nonlinear feedback.
- Stable dynamic regimes distinct from equilibrium.

They are central to:

- Biological rhythms.
- Economic cycles.
- Electrical oscillators.
- Predator–prey dynamics.

They emerge when feedback and nonlinearity combine.

Structural Compression

Invariant

A limit cycle is an isolated periodic orbit:

$$\gamma(t+T) = \gamma(t), \quad \Gamma \text{ isolated.}$$

Mechanism

Nonlinear feedback injects and dissipates energy depending on amplitude.

Cross-System Echo

- Control Theory: oscillatory closed-loop behavior.
- Biology: circadian rhythms.
- Economics: endogenous business cycles.
- Physics: nonlinear oscillators.

Systems Thinking · Module 2 — Dynamical Systems · Node 2.5
Concepts: dynamical-systems, limit-cycle, bifurcation, phase-portrait
Prerequisites: 2.2, 2.4
Enables: 3.1, 5.3

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