Module 3 — Stability And Control

Systems Thinking

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Lyapunov Stability

Position in Knowledge Graph

This node formalizes stability of equilibria using Lyapunov's method.

It builds on: - Fixed points (2.3) - Limit cycles and local dynamics (2.5)

It prepares for: - Linear system analysis (3.2) - Controllability (3.3)

Lyapunov stability provides a non-spectral method for proving stability.

Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}^n$ and consider

$$\dot{x} = f(x), \quad f : \mathcal{X} \to \mathbb{R}^n,$$

with f continuously differentiable.

Let $x^* \in \mathcal{X}$ satisfy

$$f(x^*) = 0.$$

Definition (Lyapunov Stability)

The equilibrium x^* is **Lyapunov stable** if:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x(0) - x^*\| < \delta \quad \Rightarrow \quad \|x(t) - x^*\| < \varepsilon \quad \forall t \geq 0.$$

Definition (Asymptotic Stability)

The equilibrium x^* is **asymptotically stable** if:

1. It is Lyapunov stable.
2. There exists $\delta_0 > 0$ such that

$$\|x(0) - x^*\| < \delta_0 \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = x^*.$$

Definition (Lyapunov Function)

A continuously differentiable function

$$V : \mathcal{X} \to \mathbb{R}$$

is a Lyapunov function near x^* if:

1. $V(x^*) = 0$.
2. $V(x) > 0$ for $x \neq x^*$ in a neighborhood.
3. The derivative along trajectories satisfies:

$$\dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0.$$

If

$$\dot{V}(x) < 0 \quad \text{for } x \neq x^*,$$

then the equilibrium is asymptotically stable.

Assumptions and Conditions

1. f is continuously differentiable.
2. Solutions exist and are unique.
3. Stability is defined locally unless stated otherwise.
4. Lyapunov function must be positive definite near x^*.

No linearity assumption is required.

Proof or Mathematical Development

Let V satisfy:

$$V(x) > 0 \text{ for } x \neq x^*, \quad V(x^*) = 0, \quad \dot{V}(x) \leq 0.$$

Then along any trajectory:

$$\frac{d}{dt} V(x(t)) = \nabla V(x(t)) \cdot f(x(t)) \leq 0.$$

Thus:

$$V(x(t)) \leq V(x(0)) \quad \forall t \geq 0.$$

Hence V is nonincreasing along trajectories.

For sufficiently small initial conditions, trajectories remain inside sublevel sets:

$$\Omega_c = \{x : V(x) \le c\}.$$

Since V is positive definite, sublevel sets near x^* are bounded.

Thus solutions cannot escape an arbitrarily small neighborhood, proving Lyapunov stability.

If

$$\dot{V}(x) < 0,$$

then $V(x(t))$ strictly decreases unless $x = x^*$, implying convergence.

Structural Role

Lyapunov stability:

- Generalizes linear stability.
- Avoids explicit eigenvalue computation.
- Applies to nonlinear systems.
- Connects energy-like functions to stability.

It is the primary analytic method for nonlinear feedback systems.

Structural Compression

Invariant

Stability is certified by a positive definite function with nonincreasing derivative:

$$\dot{V}(x) \le 0.$$

Mechanism

Energy-like scalar function decreases along trajectories.

Cross-System Echo

- Physics: energy dissipation.
- Control Theory: Lyapunov candidate functions.
- Economics: potential functions in equilibrium analysis.
- Biology: homeostatic energy regulation.

Linear System Analysis

Position in Knowledge Graph

This node develops explicit stability and trajectory analysis for linear time-invariant (LTI) systems.

It builds on: - Lyapunov Stability (3.1)

It prepares for: - Controllability (3.3) - Robustness (3.5)

Linear systems provide exact analytic criteria via matrix structure.

Formal Statement / Definition / Construction

Consider the continuous-time linear system:

$$\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{n\times n}, \quad x(t) \in \mathbb{R}^n.$$

The solution with initial condition $x(0) = x_0$ is:

$$x(t) = e^{At}x_0,$$

where the matrix exponential is defined by:

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}.$$

Assumptions and Conditions

1. A is constant (time-invariant).
2. The state space is $\mathbb{R}^n$.
3. Existence and uniqueness follow from linearity.
4. Stability is determined by the spectrum of A.

No symmetry or diagonalizability assumption is imposed.

Proof or Mathematical Development

Solution via Matrix Exponential

Differentiate:

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Thus:

$$\frac{d}{dt}(e^{At}x_0) = Ae^{At}x_0 = Ax(t),$$

so $x(t) = e^{At}x_0$ solves the system.

Stability Criterion

Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of A.

If A is diagonalizable:

$$A = P\Lambda P^{-1},$$

then

$$e^{At} = Pe^{\Lambda t}P^{-1},$$

where

$$e^{\Lambda t} = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}).$$

Hence:

- If
  $$\mathrm{Re}(\lambda_i) < 0 \quad \forall i,$$
  then
  $$\|x(t)\| \to 0 \quad \text{as } t \to \infty,$$
  and the equilibrium at 0 is asymptotically stable.
- If any
  $$\mathrm{Re}(\lambda_i) > 0,$$
  then solutions grow exponentially.
- If all
  $$\mathrm{Re}(\lambda_i) \leq 0$$
  and some eigenvalue has zero real part, further analysis is required.

Lyapunov Interpretation

If all eigenvalues satisfy:

$$\mathrm{Re}(\lambda_i) < 0,$$

then there exists a symmetric positive definite matrix P such that

$$A^\top P + PA = -Q, \quad Q > 0.$$

Then

$$V(x) = x^\top Px$$

is a Lyapunov function, since

$$\dot{V}(x) = x^\top(A^\top P + PA)x = -x^\top Qx < 0.$$

Thus spectral stability implies existence of quadratic Lyapunov function.

Structural Role

Linear system analysis:

- Provides exact stability conditions.
- Connects eigenvalues to trajectory geometry.
- Serves as local approximation for nonlinear systems.
- Forms foundation for control design.

All local nonlinear stability reduces to linear analysis of the Jacobian.

Structural Compression

Invariant

Stability of

$$\dot{x} = Ax$$

is determined by:

$$\mathrm{Re}(\lambda_i(A)).$$

Mechanism

Matrix exponential:

$$x(t) = e^{At}x_0.$$

Cross-System Echo

- Linear Algebra: spectral decomposition.
- Control Theory: pole placement.
- Economics: linearized equilibrium dynamics.
- Physics: linearized perturbation theory.

Controllability

Position in Knowledge Graph

This node formalizes the ability to steer a linear system using external inputs.

It builds on: - Linear System Analysis (3.2)

It prepares for: - Observability (3.4) - Robustness and control design (3.5)

Controllability determines whether feedback control can reposition system state.

Formal Statement / Definition / Construction

Consider the continuous-time linear control system:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where:

- $x(t) \in \mathbb{R}^n$ is the state,
- $u(t) \in \mathbb{R}^m$ is the control input,
- $A \in \mathbb{R}^{n\times n}$,
- $B \in \mathbb{R}^{n\times m}$.

Definition (Controllability)

The system is **controllable** on interval $[0, T]$ if for any initial state $x(0) = x_0$ and any target state $x_T \in \mathbb{R}^n$, there exists a measurable control function $u(t)$ such that:

$$x(T) = x_T.$$

Theorem (Kalman Controllability Criterion)

The system is controllable if and only if the controllability matrix:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

has full rank:

$$\text{rank}(\mathcal{C}) = n.$$

Assumptions and Conditions

1. A and B are constant matrices.
2. State space is $\mathbb{R}^n$.
3. Inputs $u(t)$ are piecewise continuous.
4. Time horizon $T > 0$ is finite.

Proof or Mathematical Development

Step 1: Variation of Constants Formula

The solution is:

$$x(T) = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau.$$

Define the reachable set from $x_0 = 0$:

$$\mathcal{R}(T) = \left\{ \int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau \right\}.$$

The system is controllable if:

$$\mathcal{R}(T) = \mathbb{R}^n.$$

Step 2: Taylor Expansion of Matrix Exponential

Expand:

$$e^{A(T-\tau)} = \sum_{k=0}^{\infty} \frac{A^k(T-\tau)^k}{k!}.$$

Substitute into the integral:

$$x(T) = \sum_{k=0}^{\infty} A^k B \int_0^T \frac{(T-\tau)^k}{k!} u(\tau)\,d\tau.$$

Thus reachable states lie in the span of:

$$\{B, AB, A^2B, \dots\}.$$

By Cayley–Hamilton theorem, powers beyond A^{n-1} are linearly dependent.

Hence reachable states are contained in the column space of:

$$\mathcal{C} = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}.$$

If this matrix has rank n, its column space is $\mathbb{R}^n$, so any state is reachable.

Conversely, if rank $< n$, reachable states lie in a proper subspace, so full controllability fails.

Structural Role

Controllability:

- Determines whether external inputs can reposition system state.
- Provides algebraic test for steering capability.
- Is prerequisite for pole placement and feedback design.
- Distinguishes passive dynamics from steerable systems.

Without controllability, stabilization via feedback may be impossible.

Structural Compression

Invariant

Controllability $\Leftrightarrow$

$$\operatorname{rank}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n.$$

Mechanism

Reachable states span the column space generated by successive action of A on B.

Cross-System Echo

- Linear Algebra: span and rank conditions.
- Control Theory: state steering and pole placement.
- Economics: policy instruments influencing macro states.
- Network Systems: actuator placement in distributed systems.

Observability

Position in Knowledge Graph

This node formalizes whether the internal state of a linear system can be reconstructed from output measurements.

It builds on: - Linear System Analysis (3.2) - Controllability (3.3)

It prepares for: - Robustness and feedback design (3.5)

Observability is the dual structural property to controllability.

Formal Statement / Definition / Construction

Consider the linear time-invariant system:

$$\dot{x}(t) = Ax(t),$$

$$y(t) = Cx(t),$$

where:

- $x(t) \in \mathbb{R}^n$ is the state,
- $y(t) \in \mathbb{R}^p$ is the measured output,
- $A \in \mathbb{R}^{n \times n}$,
- $C \in \mathbb{R}^{p \times n}$.

Definition (Observability)

The system is **observable** on interval $[0, T]$ if for any two initial states $x_0 \neq x_1$,

$$y(t; x_0) = y(t; x_1) \quad \forall t \in [0, T]$$

implies

$$x_0 = x_1.$$

Equivalently, the initial state can be uniquely determined from the output trajectory.

Theorem (Kalman Observability Criterion)

Define the observability matrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

The system is observable if and only if:

$$\text{rank}(\mathcal{O}) = n.$$

Assumptions and Conditions

1. A and C are constant matrices.
2. State space is $\mathbb{R}^n$.
3. Solutions are unique.
4. Output measurements are exact (no noise considered here).

Proof or Mathematical Development

Step 1: Output Expression

Solution of the system:

$$x(t) = e^{At}x_0.$$

Output:

$$y(t) = Ce^{At}x_0.$$

If two states produce identical outputs:

$$Ce^{At}(x_0 - x_1) = 0 \quad \forall t.$$

Define $z = x_0 - x_1$. Then:

$$Ce^{At}z = 0 \quad \forall t.$$

Step 2: Taylor Expansion

Expand the matrix exponential:

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

Substitute:

$$Ce^{At}z = \sum_{k=0}^{\infty} \frac{CA^k z}{k!} t^k.$$

For this expression to be zero for all t, each coefficient must vanish:

$$CA^k z = 0 \quad \text{for } k = 0, \dots, n-1.$$

Thus:

$$\mathcal{O}z = 0.$$

Observability requires that the only solution is $z = 0$.

Hence:

$$\operatorname{rank}(\mathcal{O}) = n.$$

Structural Role

Observability:

- Determines whether internal states are inferable from outputs.
- Is dual to controllability.
- Enables state estimation.
- Is necessary for feedback based on measurements.

Without observability, hidden modes may destabilize control systems.

Structural Compression

Invariant

Observability $\Leftrightarrow$

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$

Mechanism

Output trajectory encodes state via successive action of A filtered by C.

Cross-System Echo

- Linear Algebra: kernel and rank conditions.
- Control Theory: state estimation.
- Economics: latent variable identification.
- Network Systems: sensor placement for state reconstruction.

Robustness

Position in Knowledge Graph

This node formalizes stability and performance preservation under perturbations.

It builds on: - Lyapunov Stability (3.1) - Linear System Analysis (3.2) - Controllability (3.3) - Observability (3.4)

It prepares for: - Network dynamics (4.1) - Constraint and governance architectures (9.1)

Robustness determines whether feedback structure tolerates uncertainty.

Formal Statement / Definition / Construction

Consider the linear system:

$$\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{n \times n}.$$

Let the system be subject to perturbation:

$$\dot{x}(t) = (A + \Delta A)x(t),$$

where $\Delta A \in \mathbb{R}^{n \times n}$ is a perturbation matrix.

Definition (Robust Stability)

The system is **robustly stable** with respect to perturbations ΔA in a set $\mathcal{D} \subseteq \mathbb{R}^{n \times n}$ if:

For all $\Delta A \in \mathcal{D}$,

$$\mathrm{Re}(\lambda_i(A + \Delta A)) < 0 \quad \forall i.$$

Norm-Bounded Perturbations

Suppose

$$\|\Delta A\| \leq \varepsilon,$$

for some matrix norm.

The system is robustly stable for perturbations of size ε if:

$$\sup_{\|\Delta A\| \leq \varepsilon} \max_i \mathrm{Re}(\lambda_i(A + \Delta A)) < 0.$$

Assumptions and Conditions

1. A is Hurwitz:
$$\mathrm{Re}(\lambda_i(A)) < 0.$$
2. Perturbations are bounded in norm.
3. The matrix norm is induced by a vector norm.
4. Spectral continuity holds under small perturbations.

No assumption of diagonalizability is required.

Proof or Mathematical Development

Spectral Perturbation Bound

Let $\lambda_i(A)$ be eigenvalues of A.

For sufficiently small perturbations:

$$|\lambda_i(A + \Delta A) - \lambda_i(A)| \leq \|\Delta A\|.$$

If the spectral abscissa is defined as:

$$\alpha(A) = \max_i \mathrm{Re}(\lambda_i(A)),$$

and A is stable:

$$\alpha(A) < 0,$$

then robustness margin is determined by:

$$\varepsilon^* = -\alpha(A).$$

If

$$\|\Delta A\| < \varepsilon^*,$$

then eigenvalues remain in the left half-plane.

Lyapunov Characterization

If there exists $P > 0$ satisfying:

$$A^\top P + PA = -Q, \quad Q > 0,$$

then for perturbed system:

$$(A + \Delta A)^\top P + P(A + \Delta A) = -Q + (\Delta A^\top P + P\Delta A).$$

If

$$
\|\Delta A^\top P + P\Delta A\| < \lambda_{\min}(Q),
$$

then the perturbed system remains stable.

Thus quadratic Lyapunov functions provide robustness margins.

Structural Role

Robustness:

- Measures tolerance to uncertainty.
- Quantifies stability margins.
- Connects eigenvalue placement to perturbation resilience.
- Underlies feedback controller design.

It extends stability from exact models to uncertain systems.

Structural Compression

Invariant

Robust stability requires:

$$
\mathrm{Re}(\lambda_i(A + \Delta A)) < 0 \quad \forall \Delta A \in \mathcal{D}.
$$

Mechanism

Eigenvalue continuity and Lyapunov inequality bound perturbation effects.

Cross-System Echo

- Control Theory: gain and phase margins.
- Economics: resilience to shocks.
- Biology: homeostatic robustness.
- Network Systems: tolerance to link failure.