# Lyapunov Stability

Systems Thinking · Module 3 — Stability And Control

Node 3.1

## Position in Knowledge Graph

This node formalizes stability of equilibria using Lyapunov's method.

It builds on: - Fixed points (2.3) - Limit cycles and local dynamics (2.5)

It prepares for: - Linear system analysis (3.2) - Controllability (3.3)

Lyapunov stability provides a non-spectral method for proving stability.

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## Formal Statement / Definition / Construction

Let $\mathcal{X} \subseteq \mathbb{R}^n$ and consider

$$\dot{x} = f(x), \quad f : \mathcal{X} \to \mathbb{R}^n,$$

with $f$ continuously differentiable.

Let $x^* \in \mathcal{X}$ satisfy

$$f(x^*) = 0.$$

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### Definition (Lyapunov Stability)

The equilibrium $x^*$ is **Lyapunov stable** if:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x(0) - x^*\| < \delta \quad \Rightarrow \quad \|x(t) - x^*\| < \varepsilon \quad \forall t \geq 0.$$

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### Definition (Asymptotic Stability)

The equilibrium $x^*$ is **asymptotically stable** if:

1. It is Lyapunov stable.
2. There exists $\delta_0 > 0$ such that

$$\|x(0) - x^*\| < \delta_0 \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = x^*.$$

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**Definition (Lyapunov Function)**

A continuously differentiable function

$$V : \mathcal{X} \to \mathbb{R}$$

is a Lyapunov function near $x^*$ if:

1. $V(x^*) = 0$.
2. $V(x) > 0$ for $x \neq x^*$ in a neighborhood.
3. The derivative along trajectories satisfies:

$$\dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0.$$

If

$$\dot{V}(x) < 0 \quad \text{for } x \neq x^*,$$

then the equilibrium is asymptotically stable.

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## Assumptions and Conditions

1. $f$ is continuously differentiable.
2. Solutions exist and are unique.
3. Stability is defined locally unless stated otherwise.
4. Lyapunov function must be positive definite near $x^*$.

No linearity assumption is required.

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## Proof or Mathematical Development

Let $V$ satisfy:

$$V(x) > 0 \text{ for } x \neq x^*, \quad V(x^*) = 0, \quad \dot{V}(x) \leq 0.$$

Then along any trajectory:

$$\frac{d}{dt} V(x(t)) = \nabla V(x(t)) \cdot f(x(t)) \leq 0.$$

Thus:

$$V(x(t)) \leq V(x(0)) \quad \forall t \geq 0.$$

Hence $V$ is nonincreasing along trajectories.

For sufficiently small initial conditions, trajectories remain inside sublevel sets:

$$\Omega_c = \{x : V(x) \leq c\}.$$

Since $V$ is positive definite, sublevel sets near $x^*$ are bounded.

Thus solutions cannot escape an arbitrarily small neighborhood, proving Lyapunov stability.

If

$$\dot{V}(x) < 0,$$

then $V(x(t))$ strictly decreases unless $x = x^*$, implying convergence.

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## Structural Role

Lyapunov stability:

- Generalizes linear stability.
- Avoids explicit eigenvalue computation.
- Applies to nonlinear systems.
- Connects energy-like functions to stability.

It is the primary analytic method for nonlinear feedback systems.

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## Structural Compression

### Invariant

Stability is certified by a positive definite function with nonincreasing derivative:

$$\dot{V}(x) \leq 0.$$

### Mechanism

Energy-like scalar function decreases along trajectories.

### Cross-System Echo

- Physics: energy dissipation.
- Control Theory: Lyapunov candidate functions.
- Economics: potential functions in equilibrium analysis.
- Biology: homeostatic energy regulation.

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*Systems Thinking · Module 3 — Stability And Control · Node 3.1*
**Concepts:** stability, dynamical-systems, fixed-point
**Prerequisites:** 2.3, 2.5
**Enables:** 3.2, 3.3

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