Linear System Analysis

Systems Thinking · Module 3 — Stability And Control

Node 3.2

Position in Knowledge Graph

This node develops explicit stability and trajectory analysis for linear time-invariant (LTI) systems.

It builds on: - Lyapunov Stability (3.1)

It prepares for: - Controllability (3.3) - Robustness (3.5)

Linear systems provide exact analytic criteria via matrix structure.

Formal Statement / Definition / Construction

Consider the continuous-time linear system:

$$\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{n \times n}, \quad x(t) \in \mathbb{R}^n.$$

The solution with initial condition $x(0) = x_0$ is:

$$x(t) = e^{At} x_0,$$

where the matrix exponential is defined by:

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}.$$

Assumptions and Conditions

1. A is constant (time-invariant).
2. The state space is $\mathbb{R}^n$.
3. Existence and uniqueness follow from linearity.
4. Stability is determined by the spectrum of A.

No symmetry or diagonalizability assumption is imposed.

Proof or Mathematical Development

Solution via Matrix Exponential

Differentiate:

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Thus:

$$\frac{d}{dt}(e^{At}x_0) = Ae^{At}x_0 = Ax(t),$$

so $x(t) = e^{At}x_0$ solves the system.

Stability Criterion

Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of A.

If A is diagonalizable:

$$A = P\Lambda P^{-1},$$

then

$$e^{At} = Pe^{\Lambda t}P^{-1},$$

where

$$e^{\Lambda t} = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}).$$

Hence:

- If
  $$\mathrm{Re}(\lambda_i) < 0 \quad \forall i,$$
  then
  $$\|x(t)\| \to 0 \quad \text{as } t \to \infty,$$
  and the equilibrium at 0 is asymptotically stable.
- If any
  $$\mathrm{Re}(\lambda_i) > 0,$$
  then solutions grow exponentially.
- If all
  $$\mathrm{Re}(\lambda_i) \leq 0$$
  and some eigenvalue has zero real part, further analysis is required.

Lyapunov Interpretation

If all eigenvalues satisfy:

$$\mathrm{Re}(\lambda_i) < 0,$$

then there exists a symmetric positive definite matrix P such that

$$A^\top P + PA = -Q, \quad Q > 0.$$

Then

$$V(x) = x^\top P x$$

is a Lyapunov function, since

$$\dot{V}(x) = x^\top (A^\top P + PA)x = -x^\top Q x < 0.$$

Thus spectral stability implies existence of quadratic Lyapunov function.

Structural Role

Linear system analysis:

- Provides exact stability conditions.
- Connects eigenvalues to trajectory geometry.
- Serves as local approximation for nonlinear systems.
- Forms foundation for control design.

All local nonlinear stability reduces to linear analysis of the Jacobian.

Structural Compression

Invariant

Stability of

$$\dot{x} = Ax$$

is determined by:

$$\mathrm{Re}(\lambda_i(A)).$$

Mechanism

Matrix exponential:

$$x(t) = e^{At} x_0.$$

Cross-System Echo

- Linear Algebra: spectral decomposition.
- Control Theory: pole placement.
- Economics: linearized equilibrium dynamics.
- Physics: linearized perturbation theory.

Systems Thinking · Module 3 — Stability And Control · Node 3.2
Concepts: stability, linear-operators, eigenvalue-decomposition, matrix-representation
Prerequisites: 3.1
Enables: 3.3, 3.5

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