Controllability

Systems Thinking · Module 3 — Stability And Control

Node 3.3

Position in Knowledge Graph

This node formalizes the ability to steer a linear system using external inputs.

It builds on: - Linear System Analysis (3.2)

It prepares for: - Observability (3.4) - Robustness and control design (3.5)

Controllability determines whether feedback control can reposition system state.

Formal Statement / Definition / Construction

Consider the continuous-time linear control system:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where:

- $x(t) \in \mathbb{R}^n$ is the state,
- $u(t) \in \mathbb{R}^m$ is the control input,
- $A \in \mathbb{R}^{n \times n}$,
- $B \in \mathbb{R}^{n \times m}$.

Definition (Controllability)

The system is **controllable** on interval $[0, T]$ if for any initial state $x(0) = x_0$ and any target state $x_T \in \mathbb{R}^n$, there exists a measurable control function $u(t)$ such that:

$$x(T) = x_T.$$

Theorem (Kalman Controllability Criterion)

The system is controllable if and only if the controllability matrix:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

has full rank:

$$\text{rank}(\mathcal{C}) = n.$$

Assumptions and Conditions

1. A and B are constant matrices.
2. State space is $\mathbb{R}^n$.
3. Inputs $u(t)$ are piecewise continuous.
4. Time horizon $T > 0$ is finite.

Proof or Mathematical Development

Step 1: Variation of Constants Formula

The solution is:

$$x(T) = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau.$$

Define the reachable set from $x_0 = 0$:

$$\mathcal{R}(T) = \left\{ \int_0^T e^{A(T-\tau)}Bu(\tau)\,d\tau \right\}.$$

The system is controllable if:

$$\mathcal{R}(T) = \mathbb{R}^n.$$

Step 2: Taylor Expansion of Matrix Exponential

Expand:

$$e^{A(T-\tau)} = \sum_{k=0}^{\infty} \frac{A^k (T-\tau)^k}{k!}.$$

Substitute into the integral:

$$x(T) = \sum_{k=0}^{\infty} A^k B \int_0^T \frac{(T-\tau)^k}{k!} u(\tau)\,d\tau.$$

Thus reachable states lie in the span of:

$$\{B, AB, A^2B, \dots\}.$$

By Cayley–Hamilton theorem, powers beyond A^{n-1} are linearly dependent.

Hence reachable states are contained in the column space of:

$$\mathcal{C} = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}.$$

If this matrix has rank n, its column space is $\mathbb{R}^n$, so any state is reachable.

Conversely, if rank $< n$, reachable states lie in a proper subspace, so full controllability fails.

Structural Role

Controllability:

- Determines whether external inputs can reposition system state.
- Provides algebraic test for steering capability.
- Is prerequisite for pole placement and feedback design.
- Distinguishes passive dynamics from steerable systems.

Without controllability, stabilization via feedback may be impossible.

Structural Compression

Invariant

Controllability $\Leftrightarrow$

$$\operatorname{rank}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n.$$

Mechanism

Reachable states span the column space generated by successive action of A on B.

Cross-System Echo

- Linear Algebra: span and rank conditions.
- Control Theory: state steering and pole placement.
- Economics: policy instruments influencing macro states.
- Network Systems: actuator placement in distributed systems.

Systems Thinking · Module 3 — Stability And Control · Node 3.3
Concepts: controllability, control-theory, matrix-representation, linear-operators
Prerequisites: 3.2
Enables: 3.4, 3.5

hbar.university — own.your.cognition