Observability

Systems Thinking · Module 3 — Stability And Control

Node 3.4

Position in Knowledge Graph

This node formalizes whether the internal state of a linear system can be reconstructed from output measurements.

It builds on: - Linear System Analysis (3.2) - Controllability (3.3)

It prepares for: - Robustness and feedback design (3.5)

Observability is the dual structural property to controllability.

Formal Statement / Definition / Construction

Consider the linear time-invariant system:

$$\dot{x}(t) = Ax(t),$$

$$y(t) = Cx(t),$$

where:

- $x(t) \in \mathbb{R}^n$ is the state,
- $y(t) \in \mathbb{R}^p$ is the measured output,
- $A \in \mathbb{R}^{n \times n}$,
- $C \in \mathbb{R}^{p \times n}$.

Definition (Observability)

The system is **observable** on interval $[0, T]$ if for any two initial states $x_0 \neq x_1$,

$$y(t; x_0) = y(t; x_1) \quad \forall t \in [0, T]$$

implies

$$x_0 = x_1.$$

Equivalently, the initial state can be uniquely determined from the output trajectory.

Theorem (Kalman Observability Criterion)

Define the observability matrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

The system is observable if and only if:

$$\operatorname{rank}(\mathcal{O}) = n.$$

Assumptions and Conditions

1. A and C are constant matrices.
2. State space is $\mathbb{R}^n$.
3. Solutions are unique.
4. Output measurements are exact (no noise considered here).

Proof or Mathematical Development

Step 1: Output Expression

Solution of the system:

$$x(t) = e^{At} x_0.$$

Output:

$$y(t) = C e^{At} x_0.$$

If two states produce identical outputs:

$$C e^{At}(x_0 - x_1) = 0 \quad \forall t.$$

Define $z = x_0 - x_1$. Then:

$$C e^{At} z = 0 \quad \forall t.$$

Step 2: Taylor Expansion

Expand the matrix exponential:

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

Substitute:

$$Ce^{At} z = \sum_{k=0}^{\infty} \frac{CA^k z}{k!} t^k.$$

For this expression to be zero for all t, each coefficient must vanish:

$$CA^k z = 0 \quad \text{for } k = 0, \ldots, n-1.$$

Thus:

$$\mathcal{O} z = 0.$$

Observability requires that the only solution is $z = 0$.

Hence:

$$\operatorname{rank}(\mathcal{O}) = n.$$

Structural Role

Observability:

- Determines whether internal states are inferable from outputs.
- Is dual to controllability.
- Enables state estimation.
- Is necessary for feedback based on measurements.

Without observability, hidden modes may destabilize control systems.

Structural Compression

Invariant

Observability $\Leftrightarrow$

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$

Mechanism

Output trajectory encodes state via successive action of A filtered by C.

Cross-System Echo

- Linear Algebra: kernel and rank conditions.
- Control Theory: state estimation.
- Economics: latent variable identification.
- Network Systems: sensor placement for state reconstruction.

Systems Thinking · Module 3 — Stability And Control · Node 3.4
Concepts: observability, control-theory, matrix-representation, linear-operators
Prerequisites: 3.2, 3.3
Enables: 3.5

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