Robustness

Systems Thinking · Module 3 — Stability And Control

Node 3.5

Position in Knowledge Graph

This node formalizes stability and performance preservation under perturbations.

It builds on: - Lyapunov Stability (3.1) - Linear System Analysis (3.2) - Controllability (3.3) - Observability (3.4)

It prepares for: - Network dynamics (4.1) - Constraint and governance architectures (9.1)

Robustness determines whether feedback structure tolerates uncertainty.

Formal Statement / Definition / Construction

Consider the linear system:

$$\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{n\times n}.$$

Let the system be subject to perturbation:

$$\dot{x}(t) = (A + \Delta A)x(t),$$

where $\Delta A \in \mathbb{R}^{n\times n}$ is a perturbation matrix.

Definition (Robust Stability)

The system is **robustly stable** with respect to perturbations ΔA in a set $\mathcal{D} \subseteq \mathbb{R}^{n\times n}$ if:

For all $\Delta A \in \mathcal{D}$,

$$\mathrm{Re}(\lambda_i(A + \Delta A)) < 0 \quad \forall i.$$

Norm-Bounded Perturbations

Suppose

$$\|\Delta A\| \leq \varepsilon,$$

for some matrix norm.

The system is robustly stable for perturbations of size ε if:

$$\sup_{\|\Delta A\| \le \varepsilon} \max_i \mathrm{Re}(\lambda_i(A + \Delta A)) < 0.$$

Assumptions and Conditions

1. A is Hurwitz:
$$\mathrm{Re}(\lambda_i(A)) < 0.$$
2. Perturbations are bounded in norm.
3. The matrix norm is induced by a vector norm.
4. Spectral continuity holds under small perturbations.

No assumption of diagonalizability is required.

Proof or Mathematical Development

Spectral Perturbation Bound

Let $\lambda_i(A)$ be eigenvalues of A.

For sufficiently small perturbations:

$$|\lambda_i(A + \Delta A) - \lambda_i(A)| \le \|\Delta A\|.$$

If the spectral abscissa is defined as:

$$\alpha(A) = \max_i \mathrm{Re}(\lambda_i(A)),$$

and A is stable:

$$\alpha(A) < 0,$$

then robustness margin is determined by:

$$\varepsilon^* = -\alpha(A).$$

If

$$\|\Delta A\| < \varepsilon^*,$$

then eigenvalues remain in the left half-plane.

Lyapunov Characterization

If there exists $P > 0$ satisfying:

$$A^\top P + PA = -Q, \quad Q > 0,$$

then for perturbed system:

$$(A + \Delta A)^\top P + P(A + \Delta A) = -Q + (\Delta A^\top P + P\Delta A).$$

If

$$\|\Delta A^\top P + P\Delta A\| < \lambda_{\min}(Q),$$

then the perturbed system remains stable.

Thus quadratic Lyapunov functions provide robustness margins.

Structural Role

Robustness:

- Measures tolerance to uncertainty.
- Quantifies stability margins.
- Connects eigenvalue placement to perturbation resilience.
- Underlies feedback controller design.

It extends stability from exact models to uncertain systems.

Structural Compression

Invariant

Robust stability requires:

$$\mathrm{Re}(\lambda_i(A + \Delta A)) < 0 \quad \forall \Delta A \in \mathcal{D}.$$

Mechanism

Eigenvalue continuity and Lyapunov inequality bound perturbation effects.

Cross-System Echo

- Control Theory: gain and phase margins.
- Economics: resilience to shocks.
- Biology: homeostatic robustness.
- Network Systems: tolerance to link failure.

Systems Thinking · Module 3 — Stability And Control · Node 3.5
Concepts: stability, control-theory, feedback
Prerequisites: 3.1, 3.2, 3.3, 3.4
Enables: 4.1, 9.1

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