Module 4 — Networks

Systems Thinking

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Graph Foundations

Position in Knowledge Graph

This node formalizes the mathematical structure underlying networks.

It builds on: - Causal Loop Diagrams as signed directed graphs (1.3) - Linear system representation via matrices (3.2)

It prepares for: - Centrality measures (4.2) - Network robustness (4.5)

Graphs provide the structural topology on which network dynamics evolve.

Formal Statement / Definition / Construction

Definition (Directed Graph)

A **directed graph** is a pair:

$$G = (V, E),$$

where:

- $V = \{1, \ldots, n\}$ is a finite set of nodes,
- $E \subseteq V \times V$ is a set of ordered pairs called directed edges.

If $(i, j) \in E$, there is an edge from node i to node j.

Adjacency Matrix

The **adjacency matrix** of G is:

$$A \in \mathbb{R}^{n \times n},$$

with entries:

$$A_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Weighted graphs generalize this definition:

$$A_{ij} \in \mathbb{R},$$

representing edge strength.

Degree

For a directed graph:

- Out-degree of node i:

$$d_i^{\text{out}} = \sum_{j=1}^{n} A_{ij}.$$

- In-degree of node i:

$$d_i^{\text{in}} = \sum_{j=1}^{n} A_{ji}.$$

For undirected graphs:

$$d_i = \sum_{j=1}^{n} A_{ij}.$$

Graph Laplacian (Undirected Case)

For undirected graphs:

$$L = D - A,$$

where:

$$D = \operatorname{diag}(d_1, \dots, d_n).$$

The Laplacian satisfies:

$$x^\top L x = \frac{1}{2} \sum_{i,j} A_{ij}(x_i - x_j)^2.$$

Assumptions and Conditions

1. Node set V is finite.
2. Graph may be directed or undirected.
3. Adjacency matrix fully encodes topology.
4. No dynamics are imposed yet.

Graphs describe structure, not evolution.

Proof or Mathematical Development

Path Counting

The number of length-k directed paths from node i to node j equals:

$$(A^k)_{ij}.$$

Proof:

Matrix multiplication:

$$(A^2)_{ij} = \sum_{k=1}^{n} A_{ik} A_{kj},$$

counts two-step paths.

By induction:

$$(A^k)_{ij} = \sum_{\ell_1,\dots,\ell_{k-1}} A_{i\ell_1} A_{\ell_1\ell_2} \dots A_{\ell_{k-1}j}.$$

Thus adjacency powers encode connectivity structure.

Laplacian Positivity

For undirected graphs:

$$x^\top Lx = x^\top Dx - x^\top Ax.$$

Compute:

$$x^\top Dx = \sum_i d_i x_i^2,$$

$$x^\top Ax = \sum_{i,j} A_{ij} x_i x_j.$$

Thus:

$$x^\top Lx = \frac{1}{2} \sum_{i,j} A_{ij} (x_i - x_j)^2 \geq 0.$$

Hence L is positive semidefinite.

Structural Role

Graphs:

- Encode interaction topology.
- Define information flow structure.
- Determine diffusion and synchronization patterns.
- Provide algebraic representation of network connectivity.

All network dynamics are defined relative to graph topology.

Structural Compression

Invariant

Graph structure is fully encoded by adjacency matrix:

$$G \leftrightarrow A.$$

Mechanism

Matrix powers encode path structure:

$$(A^k)_{ij} \text{ counts length-}k \text{ paths.}$$

Cross-System Echo

- Linear Algebra: spectral properties of A and L.
- Control Theory: networked system coupling matrices.
- Economics: trade and financial networks.
- Biology: gene regulatory and neural networks.

Centrality

Position in Knowledge Graph

This node formalizes quantitative measures of node importance within a network.

It builds on: - Graph Foundations (4.1)

It prepares for: - Small-world structure (4.3) - Network robustness (4.5)

Centrality translates topology into quantitative influence measures.

Formal Statement / Definition / Construction

Let $G = (V, E)$ be a graph with adjacency matrix $A \in \mathbb{R}^{n\times n}$.

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1. Degree Centrality

For node i:

$$c_i^{\text{deg}} = \sum_{j=1}^{n} A_{ij}.$$

In undirected graphs:

$$c_i^{\text{deg}} = d_i.$$

This measures immediate connectivity.

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2. Eigenvector Centrality

Define vector $c \in \mathbb{R}^n$ such that:

$$Ac = \lambda c,$$

where λ is the largest eigenvalue of A.

Node centrality is proportional to the centrality of its neighbors.

If $A \geq 0$ and is irreducible, Perron–Frobenius theorem guarantees:

- A unique largest eigenvalue $\lambda_{\text{max}} > 0$,
- A strictly positive eigenvector c.

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3. Katz Centrality

For $\alpha > 0$ sufficiently small:

$$c = (I - \alpha A)^{-1}\mathbf{1},$$

provided

$$\alpha < \frac{1}{\rho(A)},$$

where $\rho(A)$ is the spectral radius.

This accounts for all paths with exponential decay:

$$c = \sum_{k=0}^{\infty} \alpha^k A^k \mathbf{1}.$$

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4. Betweenness Centrality (Definition)

For node i:

$$c_i^{\text{btw}} = \sum_{s \neq i \neq t} \frac{\sigma_{st}(i)}{\sigma_{st}},$$

where:

- σ_{st} = number of shortest paths from s to t,
- $\sigma_{st}(i)$ = number of those paths passing through i.

Assumptions and Conditions

1. Graph has finite nodes.
2. For eigenvector centrality, assume $A \geq 0$.
3. For Perron–Frobenius guarantees, assume irreducibility.
4. For Katz centrality, require convergence condition:

$$\alpha < 1/\rho(A).$$

Proof or Mathematical Development

Eigenvector Centrality

From definition:

$$c_i = \frac{1}{\lambda} \sum_{j=1}^{n} A_{ij} c_j.$$

Thus node importance is recursively defined.

If A is nonnegative and irreducible, Perron–Frobenius ensures:

$$\lambda_{\max} = \rho(A),$$

with positive eigenvector c.

Katz Expansion

Using Neumann series:

If

$$\|\alpha A\| < 1,$$

then

$$(I - \alpha A)^{-1} = \sum_{k=0}^{\infty} (\alpha A)^k.$$

Thus:

$$c = \sum_{k=0}^{\infty} \alpha^k A^k \mathbf{1}.$$

Each term counts weighted walks of length k.

Structural Role

Centrality:

- Identifies influential nodes.
- Quantifies network hierarchy.
- Connects local structure (degree) to global structure (eigenvectors).
- Bridges topology and dynamics.

In networked dynamical systems:

$$\dot{x} = Ax,$$

dominant eigenvector centrality predicts long-term influence.

Structural Compression

Invariant

Centrality reduces to spectral or path-based aggregation of adjacency structure:

$$Ac = \lambda c.$$

Mechanism

Eigenstructure and path counting determine influence.

Cross-System Echo

- Linear Algebra: dominant eigenvectors.
- Control Theory: actuator importance.
- Economics: systemic financial importance.
- Biology: hub genes in regulatory networks.

Small-World Networks

Position in Knowledge Graph

This node formalizes the small-world property as a quantitative structural feature of graphs.

It builds on: - Graph Foundations (4.1) - Centrality and path structure (4.2)

It prepares for: - Scale-free networks (4.4) - Network robustness (4.5)

Small-world structure characterizes networks with high local clustering and short global path length.

Formal Statement / Definition / Construction

Let $G = (V, E)$ be an undirected graph with $|V| = n$.

Let adjacency matrix be $A \in \mathbb{R}^{n \times n}$.

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1. Path Length

Define the shortest-path distance $d(i, j)$ between nodes i and j.

The **average path length** is:

$$L(G) = \frac{1}{n(n-1)} \sum_{i \neq j} d(i, j).$$

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2. Clustering Coefficient

For node i, let k_i be its degree.

Let E_i be the number of edges among the neighbors of i.

The **local clustering coefficient** is:

$$C_i = \frac{2E_i}{k_i(k_i - 1)} \quad \text{for } k_i \geq 2.$$

The **average clustering coefficient** is:

$$C(G) = \frac{1}{n} \sum_{i=1}^{n} C_i.$$

Definition (Small-World Network)

A graph G is said to exhibit the **small-world property** if:

1. $L(G) \sim L_{\text{rand}}$, where L_{rand} is average path length of a random graph with same n and edge density.
2. $C(G) \gg C_{\text{rand}}$, where C_{rand} is clustering of comparable random graph.

Thus:

$$L(G) \approx O(\log n), \quad C(G) \text{ large.}$$

Assumptions and Conditions

1. Graph is finite and undirected.
2. Graph is connected (so distances are finite).
3. Random comparison graph preserves node count and mean degree.
4. Asymptotic statements refer to large n.

Proof or Mathematical Development

Random Graph Baseline

For Erdős–Rényi random graph $G(n, p)$:

- Expected average path length:

$$L_{\text{rand}} \sim \frac{\log n}{\log(np)}.$$

- Expected clustering:

$$C_{\text{rand}} \approx p.$$

Small-World Construction (Rewiring Model)

Start from a regular ring lattice with:

- Degree k,
- High clustering,
- Large path length $L \sim n/(2k)$.

Randomly rewire edges with probability β.

For small $\beta > 0$:

- Long-range shortcuts reduce $L(G)$ rapidly toward $O(\log n)$.
- Clustering $C(G)$ remains close to lattice value.

Thus small perturbation of regular lattice produces:

$$L(G) \approx L_{\text{rand}}, \quad C(G) \gg C_{\text{rand}}.$$

This demonstrates coexistence of local cohesion and global efficiency.

Structural Role

Small-world networks:

- Enable rapid information diffusion.
- Preserve local clustering.
- Appear in social, biological, and technological systems.
- Influence synchronization and epidemic spreading.

They represent intermediate topology between lattice and random graph.

Structural Compression

Invariant

Small-world property:

$$L(G) \sim O(\log n), \quad C(G) \gg C_{\mathrm{rand}}.$$

Mechanism

Sparse long-range shortcuts drastically reduce path length without destroying local clustering.

Cross-System Echo

- Physics: percolation and network transitions.
- Biology: neural connectivity structure.
- Economics: trade and communication networks.
- Control Theory: distributed consensus speed.

Scale-Free Networks

Position in Knowledge Graph

This node formalizes heavy-tailed degree structure in networks.

It builds on: - Graph Foundations (4.1) - Centrality and degree structure (4.2) - Small-world structure (4.3)

It prepares for: - Network robustness (4.5) - Emergent structure (5.1)

Scale-free networks exhibit power-law degree distributions.

Formal Statement / Definition / Construction

Let $G = (V, E)$ be an undirected graph with $|V| = n$.

Let d_i denote the degree of node i.

Define the empirical degree distribution:

$$P(k) = \frac{1}{n} \left|\{i \in V : d_i = k\}\right|.$$

Definition (Scale-Free Network)

A network is **scale-free** if its degree distribution follows a power law:

$$P(k) \sim Ck^{-\gamma}, \quad k \geq k_{\min},$$

for constants $C > 0$ and exponent $\gamma > 1$.

Equivalently, the tail probability satisfies:

$$\mathbb{P}(d \geq k) \sim k^{-(\gamma-1)}.$$

Assumptions and Conditions

1. Graph size n is large.
2. Degree distribution is asymptotic.
3. Power-law behavior holds above threshold $k_{\min}$.
4. Normalization requires $\gamma > 1$.

Finite networks only approximate ideal power laws.

Proof or Mathematical Development

Preferential Attachment Model

Construct network sequentially:

1. Start with m_0 initial nodes.
2. At each time step, add one node with $m \leq m_0$ edges.
3. Attach new edges with probability proportional to degree:

$$\mathbb{P}(i \text{ chosen}) = \frac{d_i}{\sum_j d_j}.$$

Degree Growth

Let node i be added at time t_i.

Total degree at time t is:

$$\sum_j d_j(t) = 2mt.$$

Expected degree evolution:

$$\frac{d}{dt}d_i(t) = m\frac{d_i(t)}{2mt} = \frac{d_i(t)}{2t}.$$

Solve differential equation:

$$\frac{dd_i}{dt} = \frac{d_i}{2t}.$$

Separate variables:

$$\frac{1}{d_i}dd_i = \frac{1}{2t}dt.$$

Integrate:

$$\ln d_i(t) = \frac{1}{2}\ln t + C.$$

Thus:

$$d_i(t) = C' t^{1/2}.$$

Using initial condition $d_i(t_i) = m$:

$$d_i(t) = m\left(\frac{t}{t_i}\right)^{1/2}.$$

Degree Distribution

Probability that degree exceeds k:

$$\mathbb{P}(d_i(t) \geq k) = \mathbb{P}\left(t_i \leq m^2\frac{t}{k^2}\right).$$

Since node birth times are uniform over $[0, t]$:

$$\mathbb{P}(d \geq k) \sim \frac{1}{k^2}.$$

Thus:

$$P(k) \sim k^{-3}.$$

Hence exponent $\gamma = 3$.

Structural Role

Scale-free networks:

- Contain hubs with extremely high degree.
- Exhibit heavy-tailed connectivity.
- Arise from growth and preferential attachment.
- Produce heterogeneous influence distribution.

They explain:

- Internet topology,
- Financial networks,
- Citation networks,
- Biological interaction networks.

Structural Compression

Invariant

Scale-free property:

$$P(k) \sim k^{-\gamma}.$$

Mechanism

Preferential attachment:

$$\mathbb{P}(i \text{ chosen}) \propto d_i.$$

Cross-System Echo

- Linear Algebra: dominant eigenvalue influenced by hubs.
- Economics: wealth concentration distributions.
- Biology: hub genes in metabolic networks.
- Control Theory: vulnerability concentration at hubs.

Network Robustness

Position in Knowledge Graph

This node formalizes structural resilience of networks under node or edge removal.

It builds on: - Graph Foundations (4.1) - Small-World structure (4.3) - Scale-Free structure (4.4)

It prepares for: - Emergence phenomena (5.1) - Governance and constraint architectures (9.1)

Network robustness determines systemic vulnerability and resilience.

Formal Statement / Definition / Construction

Let $G = (V, E)$ be an undirected graph with $|V| = n$.

Define the **giant component** $\mathcal{C}_{\max} \subseteq V$ as the largest connected component.

Let $f \in [0, 1]$ denote the fraction of removed nodes.

Definition (Robustness)

The network robustness function is:

$$R(f) = \frac{|\mathcal{C}_{\max}(f)|}{n},$$

where $\mathcal{C}_{\max}(f)$ is the largest component after removal of fraction f of nodes.

A network is **robust** if $R(f)$ remains large for moderate f.

Assumptions and Conditions

1. Graph is large ($n \gg 1$).
2. Node removal may be:
   - Random failure,
   - Targeted attack (highest-degree removal).
3. Connectivity is measured via largest component size.
4. Asymptotic statements assume $n \to \infty$.

Proof or Mathematical Development

Random Graph Baseline

For Erdős–Rényi graph $G(n, p)$ with mean degree:

$$\langle k \rangle = np,$$

a giant component exists if:

$$\langle k \rangle > 1.$$

Under random removal of fraction f, effective mean degree becomes:

$$\langle k \rangle_f = (1 - f)\langle k \rangle.$$

Critical threshold:

$$(1 - f_c)\langle k \rangle = 1.$$

Thus:

$$f_c = 1 - \frac{1}{\langle k \rangle}.$$

If $f > f_c$, the giant component collapses.

Scale-Free Networks

For power-law degree distribution:

$$P(k) \sim k^{-\gamma}.$$

Second moment:

$$\langle k^2 \rangle = \sum_k k^2 P(k).$$

If $2 < \gamma \leq 3$, then $\langle k^2 \rangle \to \infty$ as $n \to \infty$.

Percolation theory implies that for random failure:

$$f_c \to 1.$$

Thus scale-free networks are highly robust to random node removal.

Targeted Attack

If highest-degree nodes are removed first:

Effective connectivity drops rapidly.

In scale-free networks, removal of hubs drastically reduces connectivity, leading to small f_c.

Thus:

- Robust to random failure,
- Fragile to targeted attack.

Structural Role

Network robustness:

- Quantifies resilience under perturbation.
- Connects degree distribution to systemic risk.
- Explains vulnerability concentration in hub-dominated systems.
- Forms basis for infrastructure protection and governance design.

Topology determines systemic resilience.

Structural Compression

Invariant

Giant component persists if effective mean degree exceeds 1:

$$(1 - f)\langle k \rangle > 1.$$

Mechanism

Connectivity collapse occurs when average branching factor falls below unity.

Cross-System Echo

- Physics: percolation thresholds.
- Economics: systemic financial risk.
- Biology: metabolic network resilience.
- Control Theory: actuator/sensor failure tolerance.