Graph Foundations

Systems Thinking · Module 4 — Networks

Node 4.1

Position in Knowledge Graph

This node formalizes the mathematical structure underlying networks.

It builds on: - Causal Loop Diagrams as signed directed graphs (1.3) - Linear system representation via matrices (3.2)

It prepares for: - Centrality measures (4.2) - Network robustness (4.5)

Graphs provide the structural topology on which network dynamics evolve.

Formal Statement / Definition / Construction

Definition (Directed Graph)

A **directed graph** is a pair:

$$G = (V, E),$$

where:

- $V = \{1, \ldots, n\}$ is a finite set of nodes,
- $E \subseteq V \times V$ is a set of ordered pairs called directed edges.

If $(i, j) \in E$, there is an edge from node i to node j.

Adjacency Matrix

The **adjacency matrix** of G is:

$$A \in \mathbb{R}^{n \times n},$$

with entries:

$$A_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Weighted graphs generalize this definition:

$$A_{ij} \in \mathbb{R},$$

representing edge strength.

Degree

For a directed graph:

- Out-degree of node i:

$$d_i^{\text{out}} = \sum_{j=1}^{n} A_{ij}.$$

- In-degree of node i:

$$d_i^{\text{in}} = \sum_{j=1}^{n} A_{ji}.$$

For undirected graphs:

$$d_i = \sum_{j=1}^{n} A_{ij}.$$

Graph Laplacian (Undirected Case)

For undirected graphs:

$$L = D - A,$$

where:

$$D = \operatorname{diag}(d_1, \ldots, d_n).$$

The Laplacian satisfies:

$$x^\top L x = \frac{1}{2} \sum_{i,j} A_{ij} (x_i - x_j)^2.$$

Assumptions and Conditions

1. Node set V is finite.
2. Graph may be directed or undirected.
3. Adjacency matrix fully encodes topology.
4. No dynamics are imposed yet.

Graphs describe structure, not evolution.

Proof or Mathematical Development

Path Counting

The number of length-k directed paths from node i to node j equals:

$$(A^k)_{ij}.$$

Proof:

Matrix multiplication:

$$(A^2)_{ij} = \sum_{k=1}^{n} A_{ik} A_{kj},$$

counts two-step paths.

By induction:

$$(A^k)_{ij} = \sum_{\ell_1,\ldots,\ell_{k-1}} A_{i\ell_1} A_{\ell_1\ell_2} \ldots A_{\ell_{k-1}j}.$$

Thus adjacency powers encode connectivity structure.

Laplacian Positivity

For undirected graphs:

$$x^\top L x = x^\top D x - x^\top A x.$$

Compute:

$$x^\top D x = \sum_i d_i x_i^2,$$

$$x^\top A x = \sum_{i,j} A_{ij} x_i x_j.$$

Thus:

$$x^\top L x = \frac{1}{2} \sum_{i,j} A_{ij} (x_i - x_j)^2 \geq 0.$$

Hence L is positive semidefinite.

Structural Role

Graphs:

- Encode interaction topology.
- Define information flow structure.
- Determine diffusion and synchronization patterns.
- Provide algebraic representation of network connectivity.

All network dynamics are defined relative to graph topology.

Structural Compression

Invariant

Graph structure is fully encoded by adjacency matrix:

$$G \leftrightarrow A.$$

Mechanism

Matrix powers encode path structure:

$$(A^k)_{ij} \text{ counts length-}k\text{ paths.}$$

Cross-System Echo

- Linear Algebra: spectral properties of A and L.
- Control Theory: networked system coupling matrices.
- Economics: trade and financial networks.
- Biology: gene regulatory and neural networks.

Systems Thinking · Module 4 — Networks · Node 4.1
Concepts: graph-theory, adjacency-matrix
Prerequisites: 1.3, 3.2
Enables: 4.2, 4.5

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