Centrality

Systems Thinking · Module 4 — Networks

Node 4.2

Position in Knowledge Graph

This node formalizes quantitative measures of node importance within a network.

It builds on: - Graph Foundations (4.1)

It prepares for: - Small-world structure (4.3) - Network robustness (4.5)

Centrality translates topology into quantitative influence measures.

Formal Statement / Definition / Construction

Let $G = (V, E)$ be a graph with adjacency matrix $A \in \mathbb{R}^{n \times n}$.

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1. Degree Centrality

For node i:

$$c_i^{\text{deg}} = \sum_{j=1}^{n} A_{ij}.$$

In undirected graphs:

$$c_i^{\text{deg}} = d_i.$$

This measures immediate connectivity.

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2. Eigenvector Centrality

Define vector $c \in \mathbb{R}^n$ such that:

$$Ac = \lambda c,$$

where λ is the largest eigenvalue of A.

Node centrality is proportional to the centrality of its neighbors.

If $A \geq 0$ and is irreducible, Perron–Frobenius theorem guarantees:

- A unique largest eigenvalue $\lambda_{\max} > 0$,

- A strictly positive eigenvector c.

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3. Katz Centrality

For $\alpha > 0$ sufficiently small:

$$c = (I - \alpha A)^{-1}\mathbf{1},$$

provided

$$\alpha < \frac{1}{\rho(A)},$$

where $\rho(A)$ is the spectral radius.

This accounts for all paths with exponential decay:

$$c = \sum_{k=0}^{\infty} \alpha^k A^k \mathbf{1}.$$

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4. Betweenness Centrality (Definition)

For node i:

$$c_i^{\text{btw}} = \sum_{s \neq i \neq t} \frac{\sigma_{st}(i)}{\sigma_{st}},$$

where:

- σ_{st} = number of shortest paths from s to t,
- $\sigma_{st}(i)$ = number of those paths passing through i.

Assumptions and Conditions

1. Graph has finite nodes.
2. For eigenvector centrality, assume $A \geq 0$.
3. For Perron–Frobenius guarantees, assume irreducibility.
4. For Katz centrality, require convergence condition:

$$\alpha < 1/\rho(A).$$

Proof or Mathematical Development

Eigenvector Centrality

From definition:

$$c_i = \frac{1}{\lambda} \sum_{j=1}^{n} A_{ij} c_j.$$

Thus node importance is recursively defined.

If A is nonnegative and irreducible, Perron–Frobenius ensures:

$$\lambda_{\max} = \rho(A),$$

with positive eigenvector c.

Katz Expansion

Using Neumann series:

If

$$\|\alpha A\| < 1,$$

then

$$(I - \alpha A)^{-1} = \sum_{k=0}^{\infty} (\alpha A)^k.$$

Thus:

$$c = \sum_{k=0}^{\infty} \alpha^k A^k \mathbf{1}.$$

Each term counts weighted walks of length k.

Structural Role

Centrality:

- Identifies influential nodes.
- Quantifies network hierarchy.
- Connects local structure (degree) to global structure (eigenvectors).
- Bridges topology and dynamics.

In networked dynamical systems:

$$\dot{x} = Ax,$$

dominant eigenvector centrality predicts long-term influence.

Structural Compression

Invariant

Centrality reduces to spectral or path-based aggregation of adjacency structure:

$$Ac = \lambda c.$$

Mechanism

Eigenstructure and path counting determine influence.

Cross-System Echo

- Linear Algebra: dominant eigenvectors.
- Control Theory: actuator importance.
- Economics: systemic financial importance.
- Biology: hub genes in regulatory networks.

Systems Thinking · Module 4 — Networks · Node 4.2
Concepts: graph-theory, centrality, adjacency-matrix, eigenvalue-decomposition
Prerequisites: 4.1
Enables: 4.3, 4.5

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