Small-World Networks

Systems Thinking · Module 4 — Networks

Node 4.3

Position in Knowledge Graph

This node formalizes the small-world property as a quantitative structural feature of graphs.

It builds on: - Graph Foundations (4.1) - Centrality and path structure (4.2)

It prepares for: - Scale-free networks (4.4) - Network robustness (4.5)

Small-world structure characterizes networks with high local clustering and short global path length.

Formal Statement / Definition / Construction

Let $G = (V, E)$ be an undirected graph with $|V| = n$.

Let adjacency matrix be $A \in \mathbb{R}^{n \times n}$.

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1. Path Length

Define the shortest-path distance $d(i, j)$ between nodes i and j.

The **average path length** is:

$$L(G) = \frac{1}{n(n-1)} \sum_{i \neq j} d(i, j).$$

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2. Clustering Coefficient

For node i, let k_i be its degree.

Let E_i be the number of edges among the neighbors of i.

The **local clustering coefficient** is:

$$C_i = \frac{2E_i}{k_i(k_i - 1)} \quad \text{for } k_i \geq 2.$$

The **average clustering coefficient** is:

$$C(G) = \frac{1}{n} \sum_{i=1}^{n} C_i.$$

Definition (Small-World Network)

A graph G is said to exhibit the **small-world property** if:

1. $L(G) \sim L_{\text{rand}}$, where L_{rand} is average path length of a random graph with same n and edge density.
2. $C(G) \gg C_{\text{rand}}$, where C_{rand} is clustering of comparable random graph.

Thus:

$$L(G) \approx O(\log n), \quad C(G) \text{ large.}$$

Assumptions and Conditions

1. Graph is finite and undirected.
2. Graph is connected (so distances are finite).
3. Random comparison graph preserves node count and mean degree.
4. Asymptotic statements refer to large n.

Proof or Mathematical Development

Random Graph Baseline

For Erdős–Rényi random graph $G(n, p)$:

- Expected average path length:

$$L_{\text{rand}} \sim \frac{\log n}{\log(np)}.$$

- Expected clustering:

$$C_{\text{rand}} \approx p.$$

Small-World Construction (Rewiring Model)

Start from a regular ring lattice with:

- Degree k,
- High clustering,
- Large path length $L \sim n/(2k)$.

Randomly rewire edges with probability β.

For small $\beta > 0$:

- Long-range shortcuts reduce $L(G)$ rapidly toward $O(\log n)$.
- Clustering $C(G)$ remains close to lattice value.

Thus small perturbation of regular lattice produces:

$$L(G) \approx L_{\text{rand}}, \quad C(G) \gg C_{\text{rand}}.$$

This demonstrates coexistence of local cohesion and global efficiency.

Structural Role

Small-world networks:

- Enable rapid information diffusion.
- Preserve local clustering.
- Appear in social, biological, and technological systems.
- Influence synchronization and epidemic spreading.

They represent intermediate topology between lattice and random graph.

Structural Compression

Invariant

Small-world property:

$$L(G) \sim O(\log n), \quad C(G) \gg C_{\text{rand}}.$$

Mechanism

Sparse long-range shortcuts drastically reduce path length without destroying local clustering.

Cross-System Echo

- Physics: percolation and network transitions.
- Biology: neural connectivity structure.
- Economics: trade and communication networks.
- Control Theory: distributed consensus speed.

Systems Thinking · Module 4 — Networks · Node 4.3
Concepts: graph-theory, small-world-networks, adjacency-matrix
Prerequisites: 4.1, 4.2
Enables: 4.4, 4.5

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