Scale-Free Networks

Systems Thinking · Module 4 — Networks

Node 4.4

Position in Knowledge Graph

This node formalizes heavy-tailed degree structure in networks.

It builds on: - Graph Foundations (4.1) - Centrality and degree structure (4.2) - Small-world structure (4.3)

It prepares for: - Network robustness (4.5) - Emergent structure (5.1)

Scale-free networks exhibit power-law degree distributions.

Formal Statement / Definition / Construction

Let $G = (V, E)$ be an undirected graph with $|V| = n$.

Let d_i denote the degree of node i.

Define the empirical degree distribution:

$$P(k) = \frac{1}{n} \left| \{ i \in V : d_i = k \} \right| .$$

Definition (Scale-Free Network)

A network is **scale-free** if its degree distribution follows a power law:

$$P(k) \sim C k^{-\gamma}, \quad k \geq k_{\min},$$

for constants $C > 0$ and exponent $\gamma > 1$.

Equivalently, the tail probability satisfies:

$$\mathbb{P}(d \geq k) \sim k^{-(\gamma - 1)}.$$

Assumptions and Conditions

1. Graph size n is large.
2. Degree distribution is asymptotic.
3. Power-law behavior holds above threshold $k_{\min}$.
4. Normalization requires $\gamma > 1$.

Finite networks only approximate ideal power laws.

Proof or Mathematical Development

Preferential Attachment Model

Construct network sequentially:

1. Start with m_0 initial nodes.
2. At each time step, add one node with $m \leq m_0$ edges.
3. Attach new edges with probability proportional to degree:

$$\mathbb{P}(i \text{ chosen}) = \frac{d_i}{\sum_j d_j}.$$

Degree Growth

Let node i be added at time t_i.

Total degree at time t is:

$$\sum_j d_j(t) = 2mt.$$

Expected degree evolution:

$$\frac{d}{dt} d_i(t) = m \frac{d_i(t)}{2mt} = \frac{d_i(t)}{2t}.$$

Solve differential equation:

$$\frac{dd_i}{dt} = \frac{d_i}{2t}.$$

Separate variables:

$$\frac{1}{d_i} dd_i = \frac{1}{2t} dt.$$

Integrate:

$$\ln d_i(t) = \frac{1}{2} \ln t + C.$$

Thus:

$$d_i(t) = C' t^{1/2}.$$

Using initial condition $d_i(t_i) = m$:

$$d_i(t) = m \left(\frac{t}{t_i} \right)^{1/2}.$$

Degree Distribution

Probability that degree exceeds k:

$$\mathbb{P}(d_i(t) \geq k) = \mathbb{P}\left(t_i \leq m^2 \frac{t}{k^2}\right).$$

Since node birth times are uniform over $[0, t]$:

$$\mathbb{P}(d \geq k) \sim \frac{1}{k^2}.$$

Thus:

$$P(k) \sim k^{-3}.$$

Hence exponent $\gamma = 3$.

Structural Role

Scale-free networks:

- Contain hubs with extremely high degree.
- Exhibit heavy-tailed connectivity.
- Arise from growth and preferential attachment.
- Produce heterogeneous influence distribution.

They explain:

- Internet topology,
- Financial networks,
- Citation networks,
- Biological interaction networks.

Structural Compression

Invariant

Scale-free property:

$$P(k) \sim k^{-\gamma}.$$

Mechanism

Preferential attachment:

$$\mathbb{P}(i \text{ chosen}) \propto d_i.$$

Cross-System Echo

- Linear Algebra: dominant eigenvalue influenced by hubs.
- Economics: wealth concentration distributions.
- Biology: hub genes in metabolic networks.

- Control Theory: vulnerability concentration at hubs.

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Concepts: graph-theory, scale-free-networks, centrality
Prerequisites: 4.1, 4.2, 4.3
Enables: 4.5, 5.1