Information-Theoretic Complexity

Systems Thinking · Module 6 — Complexity

Node 6.3

Position in Knowledge Graph

This node formalizes complexity as uncertainty and information content of system states.

It builds on: - Computational Complexity (6.2)

It prepares for: - Kolmogorov Complexity (6.4) - Shannon Entropy (7.1)

Information-theoretic complexity quantifies structural variability in probabilistic systems.

Formal Statement / Definition / Construction

Let X be a discrete random variable with finite support $\mathcal{X}$.

Let probability mass function be:

$$p(x) = \mathbb{P}(X = x).$$

Definition (Shannon Entropy)

The entropy of X is:

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x).$$

Entropy measures average information content.

Joint Entropy

For random variables X, Y:

$$H(X,Y) = -\sum_{x,y} p(x,y) \log p(x,y).$$

Conditional Entropy

$$H(X|Y) = H(X,Y) - H(Y).$$

Mutual Information

$$I(X;Y) = H(X) - H(X|Y).$$

Equivalently:

$$I(X;Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)}.$$

Assumptions and Conditions

1. Random variables are discrete.
2. Logarithm base fixed (base 2 for bits or natural log).
3. $0 \log 0$ defined as 0 by continuity.
4. Entropy is finite if support finite.

Continuous entropy requires differential entropy (not treated here).

Proof or Mathematical Development

Non-Negativity of Mutual Information

Define:

$$D(p(x,y) \| p(x)p(y)) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)}.$$

This is Kullback–Leibler divergence.

Since KL divergence satisfies:

$$D(p\|q) \geq 0,$$

we have:

$$I(X;Y) \geq 0.$$

Equality holds iff:

$$p(x,y) = p(x)p(y),$$

i.e., independence.

Entropy Bounds

If $|\mathcal{X}| = m$, then:

$$0 \leq H(X) \leq \log m.$$

Proof:

Maximum occurs for uniform distribution:

$$p(x) = \frac{1}{m}.$$

Then:

$$H(X) = -m\frac{1}{m}\log\frac{1}{m} = \log m.$$

Structural Role

Information-theoretic complexity:

- Measures unpredictability.
- Quantifies macroscopic disorder.
- Links probabilistic structure to system variability.
- Connects emergence to statistical description.

Entropy provides scalar measure of state-space dispersion.

Structural Compression

Invariant

Entropy:

$$H(X) = -\sum_x p(x)\log p(x).$$

Mechanism

Logarithmic weighting of probability mass.

Cross-System Echo

- Physics: thermodynamic entropy.
- Economics: information in markets.
- Biology: genetic diversity.
- Networks: uncertainty in signal transmission.

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Concepts: shannon-entropy, information-theory, random-variables
Prerequisites: 6.2
Enables: 6.4, 7.1

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