Kolmogorov Complexity

Systems Thinking · Module 6 — Complexity

Node 6.4

Position in Knowledge Graph

This node formalizes complexity as description length of individual objects.

It builds on: - Computational Complexity (6.2) - Information-Theoretic Complexity (6.3)

It prepares for: - Phase Transitions in Complex Systems (6.5)

Kolmogorov complexity shifts from probabilistic ensembles to individual strings.

Formal Statement / Definition / Construction

Let $\Sigma = \{0, 1\}$.

Fix a universal prefix Turing machine U.

For a finite binary string $x \in \Sigma^*$, the **Kolmogorov complexity** of x is:

$$K_U(x) = \min\{|p| : U(p) = x\},$$

where:

- $p \in \Sigma^*$ is a program,
- $|p|$ denotes length of p.

When U is fixed, write $K(x)$.

Conditional Kolmogorov Complexity

For strings x, y:

$$K(x \mid y) = \min\{|p| : U(p, y) = x\}.$$

Assumptions and Conditions

1. U is universal and prefix-free.
2. Programs are finite binary strings.
3. Complexity defined up to additive constant depending on U.
4. Machine-independence holds via invariance theorem.

No claim is made about computability.

Proof or Mathematical Development

Invariance Theorem

Let U_1 and U_2 be universal prefix machines.

Then there exists constant c such that:

$$|K_{U_1}(x) - K_{U_2}(x)| \leq c \quad \forall x.$$

Proof sketch:

Since U_2 is universal, there exists program q that simulates U_1.

If p is minimal program for x under U_1, then:

$$U_2(qp) = x.$$

Thus:

$$K_{U_2}(x) \leq |q| + K_{U_1}(x).$$

Symmetric argument gives reverse inequality.

Thus complexity differs only by additive constant.

Non-Computability

Suppose $K(x)$ were computable.

Define string:

$$x^* = \text{first string such that } K(x^*) > n.$$

If K computable, we could enumerate such strings and describe x^* in $O(\log n)$ bits.

Contradiction for sufficiently large n.

Thus $K(x)$ is not computable.

Structural Role

Kolmogorov complexity:

- Measures intrinsic description length.
- Captures algorithmic randomness.
- Distinguishes structured from incompressible strings.
- Removes reliance on probability distributions.

It bridges computation and information.

Structural Compression

Invariant

Kolmogorov complexity:

$$K(x) = \min_{U(p)=x} |p|.$$

Mechanism

Shortest program length on universal machine.

Cross-System Echo

- AI: model compression.
- Physics: algorithmic randomness.
- Economics: minimal description of market states.
- Networks: minimal encoding of topology.

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Concepts: computational-complexity, information-theory
Prerequisites: 6.2, 6.3
Enables: 6.5

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