Phase Transitions in Complex Systems

Systems Thinking · Module 6 — Complexity

Node 6.5

Position in Knowledge Graph

This node formalizes qualitative macroscopic transitions driven by parameter variation in large systems.

It builds on: - Critical Transitions (5.4) - Information-Theoretic Complexity (6.3) - Kolmogorov Complexity (6.4)

It prepares for: - Shannon Entropy and Information Flow (7.1)

Phase transitions connect microscopic interaction rules to discontinuous macroscopic order.

Formal Statement / Definition / Construction

Let a system consist of n interacting components with configuration space:

$$\Omega_n.$$

Let μ denote a control parameter (e.g., temperature, coupling strength).

Define probability distribution over configurations:

$$P_\mu(\omega) = \frac{1}{Z_n(\mu)} e^{-\beta H(\omega,\mu)},$$

where:

- $H(\omega, \mu)$ is an energy functional,
- $\beta > 0$ is inverse temperature,
- $Z_n(\mu)$ is partition function:

$$Z_n(\mu) = \sum_{\omega \in \Omega_n} e^{-\beta H(\omega,\mu)}.$$

Free Energy Density

Define:

$$f_n(\mu) = -\frac{1}{\beta n} \log Z_n(\mu).$$

Assume thermodynamic limit exists:

$$f(\mu) = \lim_{n\to\infty} f_n(\mu).$$

Definition (Phase Transition)

A **phase transition** occurs at $\mu = \mu^*$ if:

$$f(\mu)$$

is non-analytic at μ^*.

Equivalently, some derivative of $f(\mu)$ is discontinuous at μ^*.

Assumptions and Conditions

1. System size $n \to \infty$ (thermodynamic limit).
2. Partition function finite for each finite n.
3. Limit $f(\mu)$ exists.
4. Analyticity considered in parameter μ.

For finite n, $f_n(\mu)$ is analytic; true singularity emerges only in limit.

Proof or Mathematical Development

Analyticity for Finite Systems

For finite n, partition function:

$$Z_n(\mu) = \sum_{\omega} e^{-\beta H(\omega,\mu)}$$

is finite sum of analytic functions in μ.

Thus:

$$f_n(\mu) = -\frac{1}{\beta n} \log Z_n(\mu)$$

is analytic.

Therefore no true phase transition occurs for finite systems.

Emergence in Thermodynamic Limit

Non-analyticity may appear when:

$$f(\mu) = \lim_{n\to\infty} f_n(\mu)$$

fails to be analytic.

Example: Mean-field Ising model.

Magnetization:

$$m = \frac{1}{n} \sum_{i=1}^{n} s_i.$$

At critical temperature T_c:

- For $T > T_c$: $m = 0$.
- For $T < T_c$: $m \neq 0$.

Order parameter changes continuously, but derivative of free energy becomes discontinuous.

This is second-order phase transition.

Information-Theoretic Interpretation

Entropy:

$$S(\mu) = -\sum_{\omega} P_\mu(\omega) \log P_\mu(\omega).$$

Near critical point:

- Correlation length diverges,
- Information propagation becomes scale-free,
- System exhibits maximal susceptibility.

Thus complexity peaks near criticality.

Structural Role

Phase transitions:

- Formalize large-scale regime shifts.
- Require large system limit.
- Connect microscopic interaction to macroscopic order.
- Provide foundation for critical phenomena.

They unify:

- Emergence,
- Information,
- Nonlinear instability,
- Scaling laws.

Structural Compression

Invariant

Phase transition $\Leftrightarrow$ non-analytic free energy in limit:

$$f(\mu) = \lim_{n\to\infty} -\frac{1}{\beta n} \log Z_n(\mu).$$

Mechanism

Collective interaction induces singular behavior only in infinite-size limit.

Cross-System Echo

- Physics: thermodynamic phase changes.
- Economics: market regime shifts.
- Biology: cooperative transitions in populations.
- Networks: percolation thresholds.

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Concepts: phase-transitions, probability-space, shannon-entropy
Prerequisites: 5.4, 6.3, 6.4
Enables: 7.1

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