Channel Capacity

Systems Thinking · Module 7 — Information Flow

Node 7.2

Position in Knowledge Graph

This node formalizes the maximum reliable information transmission rate through a noisy channel.

It builds on: - Shannon Entropy (7.1)

It prepares for: - Signal Propagation (7.3) - Information Integration (7.4)

Channel capacity defines the fundamental limit of information flow.

Formal Statement / Definition / Construction

Let $X \in \mathcal{X}$ be channel input and $Y \in \mathcal{Y}$ be channel output.

A **discrete memoryless channel (DMC)** is defined by conditional probabilities:

$$p(y|x), \quad x \in \mathcal{X},\ y \in \mathcal{Y}.$$

Joint distribution:

$$p(x, y) = p(x)p(y|x).$$

Definition (Mutual Information)

$$I(X; Y) = \sum_{x,y} p(x, y) \log \frac{p(x, y)}{p(x)p(y)}.$$

Definition (Channel Capacity)

The **channel capacity** C is:

$$C = \max_{p(x)} I(X; Y).$$

The maximization is over all input distributions $p(x)$.

Assumptions and Conditions

1. Channel is memoryless.
2. Alphabets $\mathcal{X}, \mathcal{Y}$ finite.
3. Logarithm base fixed.
4. Codewords of length n allowed for transmission.

Asymptotic reliability refers to block length $n \to \infty$.

Proof or Mathematical Development

Achievability (Sketch)

For block length n:

Generate 2^{nR} codewords independently from optimal input distribution $p^*(x)$.

Using typical-set decoding:

If

$$R < I(X;Y),$$

then probability of decoding error satisfies:

$$P_e^{(n)} \to 0 \quad \text{as } n \to \infty.$$

Thus rates below capacity are achievable.

Converse

Suppose reliable transmission at rate R exists.

Let M denote transmitted message.

Then:

$$H(M) = nR.$$

Using Fano's inequality:

$$H(M|Y^n) \leq 1 + P_e^{(n)} nR.$$

If $P_e^{(n)} \to 0$,

$$H(M|Y^n) \to 0.$$

Thus:

$$nR = I(M;Y^n) + H(M|Y^n) \leq I(M;Y^n) + \epsilon_n.$$

Since channel memoryless:

$$I(M;Y^n) \le \sum_{i=1}^{n} I(X_i;Y_i) \le nC.$$

Therefore:

$$R \le C.$$

Binary Symmetric Channel Example

Let crossover probability p.

Channel capacity:

$$C = 1 - H_2(p),$$

where binary entropy:

$$H_2(p) = -p \log_2 p - (1-p)\log_2(1-p).$$

Structural Role

Channel capacity:

- Defines maximum sustainable information flow.
- Separates feasible from infeasible communication regimes.
- Connects entropy to reliability.
- Applies to biological, economic, and technological systems.

Information transfer is bounded by mutual information.

Structural Compression

Invariant

Capacity:

$$C = \max_{p(x)} I(X;Y).$$

Mechanism

Reliable transmission possible iff transmission rate $R < C$.

Cross-System Echo

- Communications: bandwidth limits.
- Biology: genetic information transmission.
- Economics: market signal distortion.
- Networks: throughput bounds.

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Concepts: channel-capacity, mutual-information, conditional-probability
Prerequisites: 7.1
Enables: 7.3, 7.4

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