Landscape Topology

Variational Quantum Algorithms · Module 3 — Cost Landscapes

Node 3.2

Position in Knowledge Graph

System: Variational Quantum Algorithms **Structural Domain:** Cost Landscapes **Node:** 3.2

Once you have a cost function (3.1), you can ask: **what does it look like as a function on parameter space?** The answer is a **landscape** — a real-valued function over a (typically high-dimensional) torus, with a topology determined jointly by the ansatz, the Hamiltonian, and the cost choice.

This node introduces the basic topological vocabulary that the rest of Module 3 will use, and characterizes the qualitative features that VQA landscapes exhibit.

It is a vocabulary node, not a theorem node. The point is to give you the words for what you're seeing when you look at a cost surface — *basin*, *ridge*, *saddle*, *plateau*, *period* — and to anchor the intuition that these features are not random but are determined by the structure of the underlying variational problem.

Domain: A Torus, Not a Vector Space

The first topological fact about VQA cost functions: their domain is a **torus**, not Euclidean space.

For an ansatz built from rotation gates $R_P(\theta)$ where each Pauli P has eigenvalues ± 1, the cost function satisfies

$$C(\boldsymbol{\theta} + 2\pi \mathbf{e}_i) = C(\boldsymbol{\theta})$$

for every parameter θ_i. This periodicity makes the natural domain $\mathbb{T}^P = (S^1)^P = [-\pi, \pi]^P$ with edges identified.

Practical consequences:

- **No "boundary" of parameter space.** Every direction wraps around. The optimizer cannot "go to infinity."
- **Parameter-norm regularization is awkward.** L2 penalties $\lambda \|\boldsymbol{\theta}\|^2$ prefer one specific origin, but on a torus there is no privileged origin — every point is equivalent topologically.
- **Multiple paths between any two points.** On a torus, you can go around the long way or the short way, and an optimizer with momentum can sometimes exploit this.
- **Topology contributes minima.** A function on a torus has, by Morse theory, a minimum number of critical points determined by the topology — at minimum, one minimum, one maximum, and (for $P \geq 2$) several saddle points. **You cannot have a torus with only one critical point.**

The torus is a standard mathematical setting for periodic optimization but is rarely emphasized in VQA papers. It quietly determines a lot of what is and isn't possible.

Five Qualitative Features

Every VQA landscape exhibits some mixture of the following features.

1. Basins

A **basin** is a region around a local minimum where the gradient points (statistically) toward the minimum. Optimizers that find a basin will descend to its bottom and stay there unless they can escape.

VQA landscapes typically have **many basins**, ranging from very shallow (small region of attraction, mild depth) to deep (large region, low energy). The structure of basins is the most important thing for an optimizer's behavior, more important than the global minimum location.

2. Ridges and Saddles

A **saddle point** is a critical point where the Hessian has both positive and negative eigenvalues. The cost decreases in some directions and increases in others. Gradient descent slows down at saddles but eventually escapes them through the descent directions; SGD can escape faster due to noise.

A **ridge** is a region of high cost separating two basins. Crossing a ridge requires either going around it (energy-conserving path) or briefly increasing the cost (energy-violating path). Most local optimizers cannot cross ridges; this is what makes them get stuck in suboptimal basins.

For typical VQA ansätze, the ratio of saddle points to local minima can be quite high — much of the "structure" of a high-dimensional landscape is saddle-shaped, and avoiding saddles is a real concern.

3. Plateaus

A **plateau** is a region where the gradient is small in all directions. Plateaus come in two flavors:

- **Topological plateaus** are regions of genuinely small gradient — flat regions of the function. They occur near critical points and in "valleys" between minima.
- **Concentration plateaus** (or barren plateaus, Module 4) are regions where the gradient is exponentially small *in expectation* — a statistical property of how the cost function is constructed, not a feature of any particular point.

Both look the same to a local optimizer: the gradient is too small to drive movement. Distinguishing them requires global information that the optimizer doesn't have.

4. Periodic Repeats

Because the domain is a torus, every feature appears infinitely many times (one per fundamental cell). A local minimum at $\boldsymbol{\theta}^*$ is also at $\boldsymbol{\theta}^* + 2\pi\mathbf{e}_i$ for every i. This means an optimizer with periodic awareness can move between fundamental cells "for free" — sometimes useful for escaping a plateau by jumping a half-period.

In practice, most optimizers are unaware of the periodic structure and treat parameter space as Euclidean. This is a small operational loss.

5. Symmetry-Equivalent Critical Points

If the ansatz has a symmetry — e.g., parameters θ_i and θ_j play interchangeable roles for some pair (i, j) — then critical points come in equivalence classes under that symmetry. A local minimum at (θ_1^*, θ_2^*) is also a local minimum at (θ_2^*, θ_1^*). This is treated in detail in node 3.4.

What VQA Landscapes Look Like in Aggregate

Combining all of the above, the typical VQA landscape has the following statistical character:

- **Many local minima.** An ansatz with P parameters typically has $O(P)$ or more local minima for chemistry-style cost functions.
- **Many more saddles.** Saddle points outnumber minima by a substantial factor in high dimensions.
- **Mostly flat in expectation.** For random initializations of expressive ansätze, the *average* gradient magnitude is small compared to the global cost range.
- **Non-trivial basin sizes.** The basins of attraction of local minima vary in size from "the global minimum with a basin filling most of parameter space" to "tiny isolated minima with negligible attraction."
- **Symmetry-related structure.** Many critical points are equivalent under exchange or sign symmetries.

This is qualitatively similar to neural network loss landscapes (as studied in deep learning theory), and many of the heuristics from that field transfer. But the *detailed* structure is determined by the ansatz design and the Hamiltonian, and it cannot be assumed from the analogy alone.

Why This Matters For Optimizer Choice

The features above interact differently with different optimizers:

- **Gradient descent** does well in smooth basins, fails on plateaus and saddles, cannot cross ridges.
- **Adam / momentum methods** can build up enough velocity to escape mild saddles and climb small ridges.
- **Population-based methods** (HOPSO, evolutionary strategies) maintain diverse points across multiple basins simultaneously.
- **Bayesian optimization** is good at navigating landscapes with few minima but expensive evaluations; struggles with high-dimensional plateau regions.
- **SPSA** is robust to noise and can escape some plateaus through random perturbation.

The thesis's argument that the choice of optimizer is the dominant variable in many VQA experiments (node 1.9) is in part a statement about how different optimizers handle the topology features above. **Two optimizers running on the same landscape can have qualitatively different trajectories because they engage with the topology differently.**

Structural Role

This node provides the **vocabulary** for the rest of Module 3. The next nodes elaborate specific features:

- 3.3: local vs global minima in detail.
- 3.4: how symmetries shape the landscape.
- 3.5: gradients and curvature, the local structure.
- 3.6: concentration of measure, the statistical structure.
- 3.7: shot noise, the stochastic structure.
- 3.8: parameter-shift rule, the gradient computation.

You'll come back to this node's vocabulary repeatedly. It is the lens.

Relations to Other Concepts

- **Morse theory** — the topology of smooth functions on manifolds. Counts critical points by index and relates them to the manifold's homology.
- **Loss landscape topology in deep learning** — analogous study of the loss landscapes of neural networks. Exhibits many of the same features.

- **Random landscape theory** — physical models of random Gaussian functions on high-dimensional spaces. Predicts saddle/minimum ratios in high dimension.
- **Spin glass landscapes** — the canonical model of a "rugged" landscape with many critical points.

Worked Example — A 2D VQE Landscape

For $H = Z_0 + Z_1 + 0.5X_0X_1$ on 2 qubits with ansatz $|\psi(\theta_1, \theta_2)\rangle = R_Y(\theta_1) \otimes R_Y(\theta_2)|00\rangle$:

$$C(\theta_1, \theta_2) = \cos\theta_1 + \cos\theta_2 + 0.5 \sin\theta_1 \sin\theta_2.$$

This landscape on the 2-torus $[-\pi, \pi]^2$ has:

- A **global minimum** near $(\theta_1, \theta_2) = (\pi - \arctan(0.25), \pi - \arctan(0.25))$ where both Z's are -1 and the XX correlation aligns.
- A **second local minimum** related by the symmetry $(\theta_1, \theta_2) \to (-\theta_1, -\theta_2)$.
- **Two saddle points** near $(0, \pi)$ and $(\pi, 0)$ where the Z's disagree.
- **A maximum** near $(0, 0)$ where everything is $+1$.
- **Periodic copies** of all of the above shifted by 2π in each direction.

You can visualize this surface as a 2D heatmap and see all of these features clearly. **This is the structure your optimizer is navigating, even when you can't visualize it.**

Visualization

Pending. A 2D heatmap of the worked example, showing minima (dark), maxima (light), saddles (mid), with periodic boundaries. Annotated with the topological vocabulary words: basin, ridge, saddle, plateau.

Exercises

1. For the 2D landscape $C(\theta_1, \theta_2) = \cos\theta_1 + \cos\theta_2$, enumerate all critical points on the fundamental cell $[-\pi, \pi]^2$ and classify them as min/max/saddle.
2. Show that on a 1-torus S^1, any continuous function must have at least one minimum and one maximum (this is a special case of Morse inequalities). Generalize to the 2-torus.
3. (VQA) Sketch how a basin's *size* differs from its *depth*, and explain why an optimizer might prefer a wide-shallow basin over a narrow-deep one even when the latter has lower energy.

Research Extensions

- **Topological data analysis of cost landscapes** — using persistent homology to characterize landscape structure across ansatz families.
- **Critical point counting for variational ansätze** — analytic results for specific ansatz classes.
- **Connectivity of basins** — when are two basins connected by descent paths in the cost-restricted manifold?
- **Mode connectivity** — borrowing ideas from neural network landscapes about connected sets of low-cost points.

Cross-links to Other Courses

- **Topology / Morse Theory** — critical points, indices, homology.
- **Statistical Physics** — random landscapes, spin glasses, replica calculations.
- **Machine Learning** — loss landscape geometry of deep networks.
- **Module 4 (Barren Plateaus)** — concentration plateaus in detail.

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Concepts: optimization-landscape, cost-function, convexity
Prerequisites: 3.1
Enables: 3.3, 3.5

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