VQA as a Control System

Variational Quantum Algorithms · Module 7 — Control Theoretic View

Node 7.1

Position in Knowledge Graph

System: Variational Quantum Algorithms **Structural Domain:** Control-Theoretic View **Node:** 7.1

This is the **opening node** of Module 7, the module that re-frames everything Modules 1-6 have built up — ansätze, costs, optimizers — in the language of **classical control theory**.

The motivation is more than aesthetic. Control theory has 70+ years of accumulated theory about steering dynamical systems with inputs, observing them with sensors, and designing feedback laws — *exactly* the same shape of problem a VQA poses. Importing the framework gives:

1. **A vocabulary** for talking about VQAs that connects to a mature engineering literature.
2. **Theorems** (controllability, observability, separation principle) that apply with minor modification.
3. **Design tools** (LQR, MPC, Pontryagin minimum principle) that can be applied to ansatz design and optimizer scheduling.
4. **A new perspective** on the open problems (barren plateaus, noise, regularization) that often illuminates them differently from the optimization-theoretic view.

The point of this module is *not* to replace the optimization view. It's to provide a second axis of understanding — one that is especially powerful for the questions Modules 9 (Hardware Noise) and 10 (VQA as Adaptive Control Systems) take up.

What A Control System Is

A **classical control system** is a dynamical system with:

- A **state** $\mathbf{x}(t) \in \mathcal{X}$ that evolves over time.
- A **control input** $\mathbf{u}(t) \in \mathcal{U}$ that the controller can choose.
- A **dynamics** $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$ that says how the state changes given the input.
- An **output** $\mathbf{y}(t) = h(\mathbf{x}, t) \in \mathcal{Y}$ that the controller can observe.
- An **objective** $J = \int L(\mathbf{x}, \mathbf{u})\, dt$ (or terminal cost) that the controller is trying to minimize.

The **control problem** is: choose $\mathbf{u}(t)$ so that J is minimized, possibly subject to constraints on $\mathbf{x}$, $\mathbf{u}$, and the observable measurements $\mathbf{y}$.

Different specializations give:

- **Open-loop control:** $\mathbf{u}(t)$ is chosen as a function of time alone, without observing the actual $\mathbf{x}(t)$.
- **Closed-loop / feedback control:** $\mathbf{u}(t) = K(\mathbf{y}(t), t)$ — the input depends on the measured output.
- **Optimal control:** the input is chosen to extremize a functional, typically using the Pontryagin minimum principle (Section 7.5).
- **Adaptive control:** the controller's parameters adapt over time in response to system behavior (Module 10).

A VQA Is A Control System

The mapping is direct:

Control concept	VQA analogue
State $\mathbf{x}$	Quantum state $\|\psi(\boldsymbol{\theta})\rangle$ (or its density matrix)
Control input $\mathbf{u}$	Circuit parameters $\boldsymbol{\theta}$ (or update steps)
Dynamics $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$	Schrödinger evolution under parameterized gates
Output $\mathbf{y}$	Measurement outcomes (estimated expectation values)
Objective J	Cost function $C(\boldsymbol{\theta}) = \langle\psi(\boldsymbol{\theta})\|H\|\psi(\boldsymbol{\theta})\rangle$
Open-loop	Pre-compiled, fixed-circuit VQA
Feedback	Parameter updates based on measurements
Optimal control	Choosing parameter trajectories to minimize cost
Adaptive control	Optimizers that change strategy based on observed behavior

This isn't an analogy — it's a translation. A VQA *is* a control system, with quantum dynamics replacing classical ODEs.

The question is what changes when you make this substitution. Most of control theory was developed for classical, deterministic, low-dimensional dynamical systems. Quantum systems are stochastic (measurement is intrinsically random), high-dimensional (2^n states), and have fundamentally different observability constraints (you collapse the state when you measure). So the import is not literal — it requires care.

Two Time Scales: The Inner And Outer Loops

A VQA has *two* dynamical loops, and the control framing makes them explicit.

Inner loop (state preparation). At fixed parameters $\boldsymbol{\theta}$, the quantum circuit evolves the state from $|0\rangle$ to $|\psi(\boldsymbol{\theta})\rangle$. The "time" variable here is the position in the circuit. Each gate is a control input that advances the state.

Outer loop (parameter optimization). The parameters $\boldsymbol{\theta}^k$ are updated iteration-by-iteration based on measurements of the cost. The "time" variable is the optimizer iteration. Each update is a control input that adjusts the *next* circuit's behavior.

These two loops have very different character:

- The **inner loop** is *unitary*, deterministic given $\boldsymbol{\theta}$, and runs in microseconds. Quantum control theory (which exists as a field) addresses it.
- The **outer loop** is *classical*, stochastic (because measurements are sampled), and runs in seconds-to-hours. This is where the optimization theory of Module 6 lives.

The control-theoretic view treats them as a *cascaded* control system: the outer loop controls the parameters, which in turn control the inner-loop dynamics, which in turn produce measurements, which feed back to the outer loop. Each level has its own controllability, observability, and stability properties.

What The Control Framing Buys You

A few specific things the control view gives that the pure-optimization view misses:

1. Controllability questions. Given a parameterized circuit, *which states can I steer the system to?* This is exactly the "reachability" question of expressivity (Module 8), but the control version connects it to a

60-year theorem stack about Lie algebras of vector fields, accessibility conditions, and the Chow-Rashevsky theorem.

2. Observability questions. Given measurement access to certain observables, *what can I learn about the underlying state?* This is the question of whether your cost function can resolve the parameter values you care about. Classical observability theory has explicit conditions (the observability matrix rank) that translate cleanly.

3. The separation principle. Classical control theory says: under linearity and Gaussian noise, you can design the state estimator separately from the feedback law and the result is optimal. A quantum analogue exists in some regimes and gives a clean architecture for adaptive VQAs.

4. Robust control. What if the dynamics aren't exactly known? Classical robust control (H_∞, μ-synthesis) gives explicit guarantees on stability margins. The VQA analogue addresses noise and miscalibration, and gives quantitative robustness statements.

5. Model predictive control. Looking ahead a few steps and optimizing over the lookahead window. Naturally fits VQA optimizer design with lookahead — the optimizer "thinks" several iterations ahead.

The control framing is *especially* useful when noise, drift, or hardware time-variation are present — exactly the regimes where the static optimization view starts to break down.

Quantum Control Vs Classical Control

Quantum control has been a field in its own right since the 1980s. It addresses the inner-loop problem: given a quantum system with controllable Hamiltonian $H(\mathbf{u}(t))$, choose $\mathbf{u}(t)$ to steer the state from one configuration to another.

Major results:

- **Controllability theorems** (Schirmer-Solomon, D'Alessandro): a quantum system is fully controllable iff the Lie algebra generated by its drift and control Hamiltonians spans $\mathfrak{su}(2^n)$. Same Lie algebra story as Section 8.1 — the dynamical Lie algebra.
- **Optimal control algorithms** (GRAPE, Krotov, CRAB): iterative methods that find time-dependent control pulses to perform target unitaries. Used for high-fidelity gate calibration.
- **Decoherence-free subspaces and dynamical decoupling**: control protocols that protect quantum states from noise.

These results are *directly relevant* to VQAs. A parameterized circuit can be viewed as a discretization of an underlying continuous control problem. Many of the GRAPE-style insights transfer.

The thesis's view: **VQA optimization is quantum control with discrete inputs, and it inherits both the strengths and the limitations of the broader quantum control field.**

Where The Mapping Breaks Down

The control-theory analogy is powerful but imperfect. A few places it strains:

1. Discrete vs continuous time. Classical control is typically continuous time ($\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$) or discrete-time with regular step sizes. A VQA is event-driven: each gate is a discrete event with no fixed time step. Translating between continuous and discrete views introduces some friction.

2. Measurement is irreversible. Classical sensors don't disturb the state; quantum measurements collapse it. So "observe and feed back" is not free — it changes the system. Continuous quantum measurement theory (Wiseman-Milburn) addresses this but is more complex than classical observation theory.

3. The cost is itself stochastic. In classical control, you typically know J exactly. In a VQA, you only have a *noisy estimate* of the cost from finitely many shots. This is a stochastic-control problem, not a deterministic one.

4. The state space is exponentially large. $|\psi\rangle \in \mathbb{C}^{2^n}$. Classical control theory works for low-dimensional state spaces. The high dimension is what makes barren plateaus possible (and what makes quantum control hard in general).

5. Two-loop structure. Classical control is usually a single loop. The VQA two-loop structure (inner unitary loop, outer classical loop) requires extending the framework.

These caveats matter, and Module 7 will address them as we go. None is fatal, but each requires the standard control concepts to be modified before they apply cleanly.

The Module 7 Plan

Module 7 will develop the control-theoretic framework in stages:

- **Node 7.2 (Observability):** what can we learn from measurements? When does the cost function have enough resolution to identify the optimal parameters?
- **Node 7.3 (Controllability):** which states can we reach? The Lie-algebraic conditions for full and partial controllability.
- **Node 7.4 (Optimal Control Theory):** the variational framework for choosing trajectories. Connects to quantum optimal control (GRAPE).
- **Node 7.5 (Pontryagin Principle):** the necessary conditions for optimality. Gives a different angle on why VQA optimization is hard.
- **Node 7.6 (Feedback Control):** closing the loop. Adaptive optimizer design and the separation principle.

Module 7 is the bridge between Module 6 (optimization dynamics) and Module 10 (adaptive control systems). The theoretical tools live here; the deployment patterns live in 10.

Structural Role

This node is the **opening framing** of Module 7. It establishes the control-theoretic vocabulary, maps VQA concepts onto control concepts, identifies the two-loop structure, and previews the rest of the module.

It is also the bridge from the optimization-theoretic view (Modules 5, 6) to the adaptive deployment view (Modules 9, 10), with control theory as the connective tissue.

Relations to Other Concepts

- **Wiener (1948)** — cybernetics, the founding text of control as an interdisciplinary field.
- **Pontryagin et al. (1962)** — the minimum principle, the variational foundation of optimal control.
- **Schirmer-Solomon (2001)** — quantum controllability theorems.
- **D'Alessandro (2007)** — *Introduction to Quantum Control and Dynamics*, the textbook.
- **Module 6 (Optimization Dynamics)** — the optimization view, complementary to this one.

Worked Example — Mapping A QAOA Onto A Control System

Consider QAOA at depth $p = 2$ on a 4-qubit MaxCut problem. The circuit is

$$|\psi(\boldsymbol{\beta}, \boldsymbol{\gamma})\rangle = e^{-i\beta_2 H_M} e^{-i\gamma_2 H_C} e^{-i\beta_1 H_M} e^{-i\gamma_1 H_C} |+\rangle^{\otimes 4},$$

where H_C is the cost Hamiltonian and H_M is the mixer.

Map onto control system.

- **State:** $|\psi(t)\rangle \in \mathcal{H} = \mathbb{C}^{16}$, evolving as the gates are applied.
- **Control input:** $\mathbf{u}(t)$ is piecewise constant, equal to β_1 on the first mixer interval, γ_1 on the first cost interval, etc. There are 4 distinct control intervals.
- **Dynamics:** Schrödinger equation $i|\dot{\psi}\rangle = H(\mathbf{u}(t))|\psi\rangle$, where $H(\mathbf{u})$ is either H_C or H_M depending on the interval.
- **Output:** measurement of H_C at the end, giving $y = \langle\psi(T)|H_C|\psi(T)\rangle$.
- **Objective:** minimize y (or equivalently maximize $-y$).

This is a **bang-bang control** problem — the control switches abruptly between two values H_C and H_M. The controller chooses how long to spend in each mode (the β_k, γ_k values) to minimize the terminal cost.

In control-theoretic language, QAOA is **discrete-time bang-bang optimal control with terminal cost**. The Pontryagin minimum principle (Section 7.5) gives explicit necessary conditions for the optimal β_k, γ_k, and these conditions match the gradient conditions from parameter-shift differentiation.

Why this matters: the mapping shows that QAOA is *not* an exotic quantum heuristic — it is a classical control problem in quantum disguise. The same theorems that govern bang-bang optimal control of classical systems govern QAOA, with appropriate modifications for the quantum state space.

Visualization

Pending. A block diagram of the VQA two-loop control system: inner loop (parameters → quantum circuit → state preparation → measurement → cost estimate), outer loop (cost estimate → optimizer → parameter update → back to inner loop). Caption: "VQA is a cascaded control system with quantum dynamics inside and classical optimization outside."

Exercises

1. For a simple 1-qubit ansatz $R_Y(\theta_2)R_X(\theta_1)|0\rangle$, identify the state, control input, dynamics, output, and objective in control-theoretic language.
2. The two-loop structure has different time scales (inner: microseconds, outer: seconds). Identify *which* control problems are most affected by this disparity. (Hint: think about adaptive control with measurement feedback.)
3. (VQA) For a quantum hardware system with a known calibration drift on the order of hours, sketch how a control-theoretic adaptive scheme would respond. Which loop (inner or outer) handles the drift?

Research Extensions

- **Quantum-classical hybrid control theory** — formalizing the two-loop structure with explicit theorems.
- **Real-time adaptive VQA controllers** — implementing feedback loops at the inner-loop time scale.
- **Robust VQA design via robust control techniques** — applying H_∞ and μ-synthesis to ansatz selection.
- **Receding-horizon (MPC) VQA optimization** — model-predictive control variants of VQA optimizers.

Cross-links to Other Courses

- **Classical Control Theory** — Wiener, Kalman, Pontryagin, the foundational literature.
- **Quantum Control (D'Alessandro)** — the existing field at the inner-loop level.
- **Dynamical Systems** — the broader framework of state-space dynamics.
- **Module 6 (Optimization Dynamics)** — the optimizer-theoretic view, complementary.
- **Module 10 (Adaptive Control Systems)** — the deployment endpoint.

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Concepts: control-theory, dynamical-systems, state-space, feedback, cost-function
Prerequisites: 2.1, 6.1
Enables: 7.2, 7.3, 7.4

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