Controllability of Parameterized Circuits

Variational Quantum Algorithms · Module 7 — Control Theoretic View

Node 7.3

Position in Knowledge Graph

System: Variational Quantum Algorithms **Structural Domain:** Control-Theoretic View **Node:** 7.3

This is the **dual** of node 7.2. Where observability asks "which states can we *learn about*?", controllability asks "which states can we *steer to*?"

For a VQA, controllability is the question of which states are *reachable* by varying the circuit parameters. This is the same question Module 8 asked under the name "expressivity," but the control-theoretic view gives it a different mathematical form (Lie algebras of vector fields, accessibility conditions, Chow-Rashevsky theorem) and a more *dynamical* flavor.

The two perspectives — expressivity (Module 8) and controllability (this node) — are equivalent in their conclusions but they are formulated differently and they each give different intuitions and proof techniques. This node walks through the controllability formulation, makes the link to expressivity explicit, and applies the controllability theorems to the practical question of ansatz design.

Classical Controllability

For a linear classical system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u},$$

the **controllability matrix** is

$$\mathcal{C} = \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix}.$$

Kalman controllability theorem: the system is controllable (i.e., from any initial state $\mathbf{x}_0$ you can reach any target state $\mathbf{x}_f$ in finite time using suitable $\mathbf{u}(t)$) if and only if $\mathcal{C}$ has full rank n.

If the rank is $r < n$, then there is an $(n-r)$-dimensional **uncontrollable subspace**: states that differ by a vector in this subspace cannot be reached from each other.

This is the linear version. For nonlinear systems, the controllability question is more delicate and uses the language of Lie algebras of vector fields.

Nonlinear Controllability And The Lie Bracket

For a nonlinear control-affine system

$$\dot{\mathbf{x}} = f_0(\mathbf{x}) + \sum_{j=1}^{m} u_j f_j(\mathbf{x}),$$

where f_0 is the **drift vector field** (autonomous dynamics with no control) and $f_1, \ldots, f_m$ are the **control vector fields**, the relevant object is the **Lie algebra generated by the vector fields**.

Lie bracket of two vector fields:

$$[f_i, f_j](\mathbf{x}) = \frac{\partial f_j}{\partial \mathbf{x}}(\mathbf{x}) f_i(\mathbf{x}) - \frac{\partial f_i}{\partial \mathbf{x}}(\mathbf{x}) f_j(\mathbf{x}).$$

The Lie bracket measures the "non-commutation" of two vector fields. Operationally, if you flow along f_i for a small time, then f_j for a small time, then $-f_i$, then $-f_j$, you don't return to the start — you arrive at a point displaced from the start by approximately $[f_i, f_j]$ times the square of the time step.

Chow-Rashevsky theorem: the system is controllable on a connected manifold if the Lie algebra generated by $\{f_0, f_1, \ldots, f_m\}$ under repeated commutation spans the tangent space at every point.

This is the generalization of Kalman's controllability theorem to nonlinear systems. The "rank" condition is replaced by a Lie-algebra spanning condition.

Quantum Controllability

The Schrödinger equation with control Hamiltonian $H(\mathbf{u}(t)) = H_0 + \sum_j u_j(t) H_j$ is

$$i|\dot{\psi}\rangle = \left(H_0 + \sum_j u_j(t) H_j \right) |\psi\rangle.$$

This is a control-affine system on the projective Hilbert space, with drift H_0 and control vector fields generated by H_j.

Theorem (Schirmer-Solomon, D'Alessandro): the system is **fully controllable on the unitary group** $U(2^n)$ (i.e., any unitary can be implemented with suitable $\mathbf{u}(t)$) if and only if the Lie algebra generated by $\{iH_0, iH_1, \ldots, iH_m\}$ under commutation is the full algebra $\mathfrak{u}(2^n)$.

This is the dynamical Lie algebra (DLA) introduced in node 8.1. The controllability theorem and the expressivity theorem are *the same theorem* in different language.

Variants:

- **State controllability (rather than unitary controllability):** the system is state-controllable if any state can be reached from any other. Weaker than full unitary controllability — requires the DLA to act *transitively* on the state space, not generate the full unitary group.
- **Operator controllability:** any unitary in some subgroup (e.g., particle-number-conserving) can be implemented. Requires the DLA to be exactly that subgroup's algebra.
- **Density-matrix controllability:** any density matrix (mixed states) can be reached. Requires *more* than full unitary controllability — you also need access to non-unitary operations (measurement, dissipation).

Controllability And Expressivity Are The Same Thing

The link is exact. For a parameterized circuit $U(\boldsymbol{\theta}) = \prod_j e^{-i\theta_j H_j}$:

- **Expressivity** = dimension of the reachable manifold = dimension of the orbit of $|0\rangle$ under the group generated by $\{e^{-i\theta H_j}\}$.
- **Controllability** = dimension of the orbit reachable under the control system with control Hamiltonians $\{H_j\}$.

These are the same dimension. The DLA $\mathfrak{g} = \mathrm{Lie}(\{iH_j\})$ governs both:

- $\dim \mathcal{R}(U) = \dim(\mathfrak{g} \cdot |0\rangle)$ (the orbit of $|0\rangle$).
- $\dim(\text{controllable set}) = \dim(\mathfrak{g} \cdot |0\rangle)$ (same).

So **the controllability theorem of quantum control theory and the expressivity theorem of VQA theory are the same theorem in different language**, both reducing to the dimension of the DLA.

This is more than a notational point — it means **70 years of quantum control theory directly applies to VQA expressivity questions**, including:

- Algorithmic methods for computing the DLA (graph-theoretic Lie closure).
- Conditions for *small* DLAs (when symmetries reduce the algebra).
- Approximate controllability theorems (when you can't reach the whole orbit but get arbitrarily close).
- Time-optimal control (the minimum time to reach a target — analogous to minimum circuit depth).

The thesis's view: **VQA practitioners should be reading the quantum control literature**, because that's where the relevant theorems live.

Reachable Set Vs Reachability In Finite Time

A subtle but important distinction:

Reachable set: the set of states you can *eventually* reach, possibly with arbitrarily long control trajectories.

Time-bounded reachable set: the set of states you can reach within a fixed time bound T.

For a fully controllable system, the reachable set is the whole projective Hilbert space. But the *time-bounded* reachable set is much smaller — it grows with T and only equals the full space in the limit $T \to \infty$.

For VQAs, "time" is replaced by **circuit depth**. The depth-bounded reachable set is the set of states reachable by a circuit of at most L layers. As L increases, this grows monotonically toward the full reachable set.

Quantitative version: for an HEA with random initialization, the depth-bounded reachable set covers an ϵ-fraction of the Hilbert space at depth $L \sim n \log(1/\epsilon)$. Below this depth, the reachable set is a sparse subset; above it, the ansatz is approximately fully expressive.

This is the bridge between the *algebraic* expressivity (DLA dimension, asymptotic) and the *practical* expressivity (what depth do I actually need?).

Approximate And Conditional Controllability

For real ansätze, exact controllability is often unattainable, but approximate or conditional versions hold:

1. Approximate controllability. The reachable set is dense in the target manifold but not equal to it. Practically, you can get *arbitrarily close* to any target state but not exactly reach it.

2. Conditional controllability. The reachable set is the full target manifold *given* certain initial conditions (e.g., starting from a specific state, or with parameters in a specific range).

3. Bounded-depth controllability. The reachable set at depth L is some subset of the full target manifold, growing with L.

4. Symmetry-restricted controllability. The reachable set is the orbit of a symmetry group. If your problem has the symmetry, this is enough.

For VQAs, **bounded-depth controllability with symmetry restriction** is the typical practical regime. You don't need to be able to reach every state in Hilbert space — you need to be able to reach every state in the symmetry-allowed sector that contains your problem's solution.

Practical Diagnostics For Controllability

How do you check if your ansatz is controllable for the problem at hand?

1. Compute the DLA. Symbolically close $\{iH_j\}$ under commutators and count the dimension. If the DLA dimension matches the expected target manifold dimension, you have full controllability.

2. Check symmetry conservation. Identify the conserved quantities (operators that commute with all H_j). The reachable set is restricted to the joint eigenspaces of these conserved quantities. Make sure your target state is in the correct eigenspace.

3. Empirical reachability test. Sample many target states from the desired distribution and check whether your ansatz can approximate each one (via local optimization). If most can be approximated, you're effectively controllable.

4. Random parameter coverage. Sample many random parameter values and check the *variety* of states produced. Low variety = limited reachable set.

These diagnostics are operational versions of the formal Lie-algebraic conditions, suitable for ansätze where exact computation of the DLA is not feasible.

Controllability And Trainability — Revisited

A key insight from Module 8 (and the Ragone et al. theorem):

Trainability is inversely proportional to controllability.

- Fully controllable ansatz (DLA = $\mathfrak{u}(2^n)$): unique answer reachable, but barren plateaus everywhere.
- Restricted-controllability ansatz (DLA = small Lie subalgebra): reachable set is smaller, but trainability is better.

The control-theoretic framing makes this very clean: **controllability and trainability are conflicting design objectives, with the DLA dimension as the single parameter that controls both**.

This is the same tradeoff that Module 8.5 stated in expressivity language. The control language gives it a slightly different flavor:

- **Expressivity language:** "How big is the reachable manifold? Bigger = harder to optimize."
- **Control language:** "Which states can the control system reach? More reachable = noisier feedback signal."

Both are correct. The choice is rhetorical.

Time-Optimal Control And Minimum Circuit Depth

A particularly nice application of control theory to VQAs: **time-optimal quantum control**.

Problem. Given a target unitary U_T and a set of control Hamiltonians $\{H_j\}$, find the **minimum time** T^* to implement U_T (within tolerance) using piecewise-constant controls.

Theorem (Khaneja-Brockett-Glaser 2001). For two-qubit systems with a specific control Hamiltonian, the minimum time has an explicit closed form in terms of the singular values of U_T. This is one of the few cases where time-optimal control of quantum systems is solved exactly.

Implication for VQAs. The minimum circuit depth to implement a target state is the discrete-time analogue. Time-optimal control gives **lower bounds** on circuit depth: no ansatz can implement the target faster than the time-optimal control trajectory.

For VQA design, this is useful: it gives you a *target depth* below which you should not bother trying. If your ansatz is at this depth or above, the question is whether the *parameter optimization* can find the right control trajectory.

Structural Role

This node is the **reachability foundation** of Module 7. It introduces classical and quantum controllability, identifies the DLA-based conditions, and establishes the equivalence between control-theoretic controllability and VQA expressivity.

It is the dual of node 7.2 (observability) and a prerequisite for node 7.4 (optimal control), which uses controllability to formulate the cost-minimization problem.

Relations to Other Concepts

- **Chow-Rashevsky theorem** — nonlinear controllability via Lie algebras.
- **Schirmer-Solomon (2001)** — quantum controllability theorem.
- **D'Alessandro (2007)** — *Introduction to Quantum Control and Dynamics.*
- **Khaneja-Brockett-Glaser (2001)** — time-optimal two-qubit control.
- **Module 8.1 (Expressivity)** — the same theorem in different language.

Worked Example — Controllability Of A 2-Qubit XX Ansatz

Consider $U(\theta_1, \theta_2, \theta_3) = e^{-i\theta_3 X_0 X_1} R_Y(\theta_2)_1 R_Y(\theta_1)_0$ — single-qubit R_Y on each qubit followed by an XX coupling.

Generators: - $H_1 = Y_0$ (generator of R_Y on qubit 0). - $H_2 = Y_1$ (generator of R_Y on qubit 1). - $H_3 = X_0 X_1$ (generator of the XX coupling).

Compute the DLA. Take Lie brackets: - $[H_1, H_3] = [Y_0, X_0 X_1] = -2i Z_0 X_1$ (new operator). - $[H_2, H_3] = [Y_1, X_0 X_1] = -2i X_0 Z_1$ (new operator). - $[H_1, [H_1, H_3]] = \ldots$ more new operators.

Continuing, the DLA closes after several steps and contains: $\{Y_0, Y_1, X_0X_1, Z_0X_1, X_0Z_1, Z_0Y_1, Y_0Z_1, Z_0Z_1, X_0, X_1, \ldots\}$.

For 2 qubits, the full Lie algebra $\mathfrak{u}(4)$ has dimension 16. The DLA of this ansatz, after closure, is exactly all of $\mathfrak{u}(4)$ — so it is **fully controllable**.

Implication. This 3-parameter ansatz (after enough repetitions) can reach any 2-qubit state. State-universal at sufficient depth.

Trainability. Because the DLA is full $\mathfrak{u}(4)$, the Ragone bound predicts gradient variance $\sim 1/16$ — small but not exponentially small. For 2 qubits, this is fine. For n qubits with similar structure, the bound becomes $\sim 1/4^n$ — barren plateau.

This is a clean instance where controllability theory gives both a positive result (universality) and a negative result (untrainability at scale) from the same calculation.

Visualization

Pending. A diagram showing two ansätze: (a) a controllable one (DLA = full algebra) with reachable set = full Hilbert space; (b) an uncontrollable one with reachable set = a low-dimensional submanifold. Caption: "Controllability is about which states you can steer to."

Exercises

1. For a 1-qubit system with control Hamiltonians $\{X, Z\}$, is it fully controllable? Compute the DLA.
2. For a 2-qubit system with control Hamiltonians $\{X_0, X_1, Z_0Z_1\}$, is it fully controllable? (Hint: this is the standard QAOA-style set.)
3. (VQA) For a 4-qubit system with only $\{Y_0, Y_1, Y_2, Y_3, Z_0Z_1, Z_1Z_2, Z_2Z_3\}$ (linear chain XX-equivalent), compute the DLA dimension. Is the ansatz fully controllable?

Research Extensions

- **Efficient DLA computation** — algorithmic methods for large ansätze.
- **Approximate controllability with explicit error bounds** — quantitative versions of "almost controllable."
- **Time-optimal VQA depth** — combining time-optimal quantum control with VQA design.
- **Controllability under noise** — how does decoherence affect the reachable set?

Cross-links to Other Courses

- **Lie Theory** — Lie algebras, Lie groups, Lie brackets.
- **Differential Geometry** — vector fields, integral curves, manifolds.
- **Classical Control Theory** — Chow-Rashevsky, Kalman controllability.
- **Module 8.1 (Expressivity Theory)** — the same content in expressivity language.
- **Module 7.4 (Optimal Control)** — uses controllability as a precondition.

Variational Quantum Algorithms · Module 7 — Control Theoretic View · Node 7.3
Concepts: controllability, parameterized-circuits, control-theory, dimensionality
Prerequisites: 7.1, 8.1
Enables: 7.4

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